The QC Relaxation: A Theoretical and Computational Study on Optimal Power Flow

Carleton Coffrin, Hassan L. Hijazi, and Pascal Van Hentenryck

Abstract—Convex relaxations of the power flow equations and, in particular, the Semi-Definite Programming (SDP) and Second-Order Cone (SOC) relaxations, have attracted significant interest in recent years. The Quadratic Convex (QC) relaxation is a departure from these relaxations in the sense that it imposes constraints to preserve stronger links between the voltage variables through convex envelopes of the polar representation. This paper is a systematic study of the QC relaxation for AC Optimal Power Flow with realistic side constraints. The main theoretical result shows that the QC relaxation is stronger than the SOC relaxation and neither dominates nor is dominated by the SDP relaxation. In addition, comprehensive computational results show that the QC relaxation may produce significant improvements in accuracy over the SOC relaxation at a reasonable computational cost, especially for networks with tight bounds on phase angle differences. The QC and SOC relaxations are also shown to be significantly faster and reliable compared to the SDP relaxation given the current state of the respective solvers.

Index Terms—Optimization Methods, Convex Quadratic Optimization, Optimal Power Flow

NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>The set of nodes in the network</td>
</tr>
<tr>
<td>E</td>
<td>The set of from edges in the network</td>
</tr>
<tr>
<td>ER</td>
<td>The set of to edges in the network</td>
</tr>
<tr>
<td>i</td>
<td>Imaginary number constant</td>
</tr>
<tr>
<td>I</td>
<td>AC current</td>
</tr>
<tr>
<td>S</td>
<td>AC power</td>
</tr>
<tr>
<td>V</td>
<td>AC voltage</td>
</tr>
<tr>
<td>Z</td>
<td>Line impedance</td>
</tr>
<tr>
<td>Y</td>
<td>Line admittance</td>
</tr>
<tr>
<td>T</td>
<td>Transformer properties</td>
</tr>
<tr>
<td>Ys</td>
<td>Bus shunt admittance</td>
</tr>
<tr>
<td>W</td>
<td>Product of two AC voltages</td>
</tr>
<tr>
<td>l</td>
<td>Current magnitude squared,</td>
</tr>
<tr>
<td>c</td>
<td>Line charging</td>
</tr>
<tr>
<td>sα</td>
<td>Line apparent power thermal limit</td>
</tr>
<tr>
<td>θΔ</td>
<td>Phase angle difference limit</td>
</tr>
<tr>
<td>Sd</td>
<td>AC power demand</td>
</tr>
<tr>
<td>Sg</td>
<td>AC power generation</td>
</tr>
<tr>
<td>c0, c1, c2</td>
<td>Generation cost coefficients</td>
</tr>
<tr>
<td>Re(·)</td>
<td>Real part of a complex number</td>
</tr>
<tr>
<td>Im(·)</td>
<td>Imaginary part of a complex number</td>
</tr>
<tr>
<td>(·)*</td>
<td>Conjugate of a complex number</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>x⁺, x⁻</td>
<td>Lower and upper bounds of x, respectively</td>
</tr>
<tr>
<td>x̃</td>
<td>Convex envelope of x</td>
</tr>
<tr>
<td>x</td>
<td>A constant value</td>
</tr>
</tbody>
</table>

CONVEX relaxations of the power flow equations have attracted significant interest in recent years. They include the Semi-Definite Programming (SDP) [1], Second-Order Cone (SOC) [2], Convex-DistFlow (CDF) [3], and the recent Quadratic Convex (QC) [4] and Moment-Based [5], [6] relaxations. Much of the excitement underlying this line of research comes from the fact that the SDP relaxation has shown to be tight on a variety of case studies [7], opening a new avenue for accurate, reliable, and efficient solutions to a variety of power system applications. Indeed, industrial-strength optimization tools (e.g., Gurobi, cplex, Mosek) are now available to solve various classes of convex optimization problems.

The relationships between the SDP, SOC, and CDF relaxations is now largely well-understood: See [8], [9] for a comprehensive overview. In particular, the SOC and CDF relaxations are known to be equivalent and the SDP relaxation is at least as strong as both of these. However, little is known about the QC relaxation which is a significant departure from these more traditional relaxations. Indeed, one of the key features of the QC relaxation is to compute convex envelopes of the polar representation of the power flow equations in the hope of preserving stronger links between the voltage variables. This contrasts with the SDP and SOC relaxations which are derived from a lift-and-project approach on the complex representation.

This paper fills this gap and provides a theoretical study of the QC relaxation as well as a comprehensive computational evaluation to compare the strengths and weaknesses of these relaxations. Our main contributions can be summarized as follows:

1) The QC relaxation is stronger than the SOC relaxation.
2) The QC relaxation neither dominates nor is dominated by the SDP relaxation.
3) Computational results on optimal power flow show that the QC relaxation may bring significant benefits in accuracy over the SOC relaxation, especially for tight bounds on phase angle differences, for a reasonable loss in efficiency.
4) The computational results also show that, with existing solvers, the SOC and QC relaxations are significantly faster and more reliable than the SDP relaxation.

The theoretical results are derived using the equivalence of two classes of second-order cone constraints (in conjunction with the power equations), which provides an alternative formulation for the QC model which is interesting in its own right.
Moreover, to the best of our knowledge, the computational results also represent the most comprehensive comparison of these convex relaxations. They are obtained for optimal power flow problems with realistic side-constraints, featuring bus shunts, line charging, and transformers.

The rest of the paper is organized as follows. Section II reviews the formulation of the AC-OPF problem from first principles and presents two equivalent formulations of this non-convex optimization problem. Section III derives the SDP, QC, and SOC relaxations. Section IV illustrates their behavior on a well-known 3-bus example. Section V presents an alternative formulation of the QC relaxation which is a convenient tool for subsequent proofs. Section VI presents the theoretical results linking the QC to the other relaxations. Section VII concludes the paper.

II. AC OPTIMAL POWER FLOW

This section reviews the specification of AC Optimal Power Flow (AC-OPF) and introduces the notations used in the paper. In the equations, constants are always in bold face. The AC power flow equations are based on complex quantities for current $I$, voltage $V$, admittance $Y$, and power $S$, which are linked by the physical properties of Kirchhoff’s Current Law (KCL), i.e.,

$$I_i^0 - I_i^d = \sum_{(i,j) \in E \cup E^R} I_{ij}$$

(1)

Ohm’s Law, i.e.,

$$I_{ij} = Y_{ij} (V_i - V_j)$$

(2)

and the definition of AC power, i.e.,

$$S_{ij} = V_i I_{ij}^*$$

(3)

Combining these three properties yields the AC Power Flow equations, i.e.,

$$S_i^0 - S_i^d = \sum_{(i,j) \in E \cup E^R} S_{ij} \forall i \in N$$

(4a)

$$S_{ij} = Y_{ij} V_i V_{ij}^* - Y_{ij}^* V_j V_{ij}^* \quad (i,j) \in E \cup E^R$$

(4b)

These non-convex nonlinear equations define how power flows in the network and are a core building block in many power system applications. However, practical applications typically include various operational side constraints on the power flow. We now review some of the most significant ones.

**Generator Capabilities:** AC generators have limitations on the amount of active and reactive power they can produce $S_i^g$, which is characterized by a generation capability curve $|S_i^g|$. Such curves typically define nonlinear convex regions which are typically approximated by boxes in AC transmission system test cases, i.e.,

$$S_i^{gl} \leq S_i^g \leq S_i^{gu} \forall i \in N$$

(5a)

**Line Thermal Limit:** AC power lines have thermal limits $|S_{ij}|$ to prevent lines from sagging and automatic protection devices from activating. These limits are typically given in Volt Amp units and constrain the apparent power flows on the lines, i.e.,

$$|S_{ij}| \leq S_{ij}^u \forall (i,j) \in E \cup E^R$$

(6)

**Bus Voltage Limits:** Voltages in AC power systems should not vary too far (typically $\pm 10\%$) from some nominal base value $|V_i^0|$. This is accomplished by putting bounds on the voltage magnitudes, i.e.,

$$v_i^l \leq |V_i| \leq v_i^u \forall i \in N$$

(7)

A variety of power flow formulations only have variables for the square of the voltage magnitude, i.e., $|V_i|^2$. In such cases, the voltage bound constrains can be incorporated via the following constraints:

$$(v_i^l)^2 \leq |V_i|^2 \leq (v_i^u)^2 \forall i \in N$$

(8)

**Phase Angle Differences:** Small phase angle differences are also a design imperative in AC power systems [10] and it has been suggested that phase angle differences are typically less than 10 degrees in practice [11]. These constraints have not typically been incorporated in AC transmission test cases [12]. However, recent work [13, 14] have observed that incorporating Phase Angle Difference (PAD) constraints, i.e.,

$$-\theta_{ij}^\Delta \leq \angle (V_i V_j^*) \leq \theta_{ij}^\Delta \forall (i,j) \in E$$

(9)

is useful in the convexification of the AC power flow equations. For simplicity, this paper assumes that the phase angle difference bounds are symmetrical and within the range $(-\pi/2, \pi/2)$, i.e.,

$$0 \leq \theta_{ij}^\Delta \leq \frac{\pi}{2} \quad (i,j) \in E$$

(10)

but the results presented here can be extended to more general cases. Observe also that the PAD constraints can be implemented as a linear relation of the real and imaginary components of $V_i V_j^*$ [13], i.e. $\forall (i,j) \in E$,

$$\tan(-\theta_{ij}^\Delta) R (V_i V_j^*) \leq \Im (V_i V_j^*) \leq \tan(\theta_{ij}^\Delta) R (V_i V_j^*)$$

(11)

The usefulness of this formulation will be apparent later in the paper.

**Other Constraints:** Other line flow constraints have been proposed, such as, active power limits and voltage difference limits [7, 14]. However, we do not consider them here since, to the best of our knowledge, test cases incorporating these constraints are not readily available.

**Objective Functions:** The last component in formulating OPF problems is an objective function. The two classic objective functions are line loss minimization, i.e.,

$$\min \sum_{i \in N} R(S_i^0)$$

(12)

and generator fuel cost minimization, i.e.,

$$\min \sum_{i \in N} c_{2i} (R(S_i^0))^2 + c_{1i} |S_i^0| + c_{0i}$$

(13)

Observe that objective (12) is a special case of objective (13) where $c_{2i} = 0, c_{1i} = 1, c_{0i} = 0 \ (i \in N)$ [15]. Hence, the rest of this paper focuses on objective (13).
Model 1 AC-OPF

variables: $S_i^q(\forall i \in N)$, $V_i(\forall i \in N)$

minimize: $\sum_{i \in N} c_{2i}(\Re(S_i^q))^2 + c_{1i}\Re(S_i^q) + c_{0i}$ (15a)

subject to:

$v_i^l \leq |V_i| \leq v_i^u \ \forall i \in N$ (15b)

$S_i^{ql} \leq S_i^q \leq S_i^{qu} \ \forall i \in N$ (15c)

$|S_{ij}| \leq s_{ij} \ \forall (i,j) \in E \cup E^R$ (15d)

$S_i^q - S_i^{ql} = \sum_{(i,j) \in E \cup E^R} S_{ij} \ \forall i \in N$ (15e)

$S_{ij} = Y_{ij}^* V_i V_j^* - Y_{ij}^* V_i V_j^* \ (i,j) \in E \cup E^R$ (15f)

$-\theta_{ij}^\Delta \leq \angle(V_i V_j^*) \leq \theta_{ij}^\Delta \ \forall (i,j) \in E$ (15g)

Model 2 AC-OPF-W

variables: $S_i^q(\forall i \in N)$, $V_i(\forall i \in N)$, $W_{ij}(\forall i, j \in N)$

minimize: $\sum_{i \in N} c_{2i}(\Re(S_i^q))^2 + c_{1i}\Re(S_i^q) + c_{0i}$ (16a)

subject to:

$W_{ij} = V_i V_j^* \ \forall i \in N, \forall j \in N$ (16b)

$(v_i^l)^2 \leq W_{ii} \leq (v_i^u)^2 \ \forall i \in N$ (16c)

$S_i^{ql} \leq S_i^q \leq S_i^{qu} \ \forall i \in N$ (16d)

$S_i^q - S_i^{ql} = \sum_{(i,j) \in E \cup E^R} S_{ij} \ \forall i \in N$ (16e)

$S_{ij} = Y_{ij}^* W_{ii} - Y_{ij}^* W_{ij} \ (i,j) \in E$ (16f)

$S_{ji} = Y_{ij}^* W_{jj} - Y_{ij}^* W_{ij} \ (i,j) \in E$ (16g)

$|S_{ij}| \leq (s_{ij}^*) \ \forall (i,j) \in E \cup E^R$ (16h)

$\tan(-\theta_{ij}^\Delta)\Re(W_{ij}) \leq \Im(W_{ij}) \leq \tan(\theta_{ij}^\Delta)\Re(W_{ij})$ (16i)

$\forall (i,j) \in E$

AC-OPF: Combining the AC power flow equations, the side constraints, and the objective function, yields the well-known AC-OPF formulation presented in Model 1. Observe that, in Model 1, the non-convexities arise solely from the product of the voltages (i.e., $V_i V_j^*$) and they can be isolated by introducing new W variables to represent the products of Vs [16, 2, 17, 18], i.e.,

$V_i V_j^* = W_{ij} \ (i,j \in N)$. (14)

Model 2 presents an equivalent version of the AC-OPF, where the W factorization has been incorporated and the only source of non-convexity is in constraint (16b). Note that this section has introduced the simplest form of the AC-OPF problem and that real-world applications feature a variety of extensions as discussed at length in [19, 20]. In practice, this non-convex nonlinear optimization problem is typically solved with numerical methods (e.g. IPM, SLP) [21, 22], which provide locally optimal solutions if they converge to a feasible point.

Model 3 The SDP Relaxation AC-OPF-W-SDP

variables: $S_i^q(\forall i \in N)$, $W_{ij}(\forall i, j \in N)$

minimize: (16a)

subject to: (16c)-(16d)

$W \geq 0$ (17a)

Model 4 The SOC Relaxation AC-OPF-W-SOC

variables: $S_i^q(\forall i \in N)$, $W_{ij}(\forall i, j \in E)$, $W_{ii}(\forall i \in N) : real$

minimize: (16a)

subject to: (16c)-(16d)

$|W_{ij}|^2 \leq W_{ii} W_{jj} \ \forall (i,j) \in E$ (18a)

III. Convex Relaxations of Optimal Power Flow

Since the AC-OPF problem is NP-Hard [23, 24] and numerical methods provide limited guarantees for determining feasibility and global optimality, significant attention has been devoted to finding convex relaxations of Model 1. Such relaxations are appealing because they are computationally efficient and may be used to:

1) bound the quality of AC-OPF solutions produced by locally optimal methods;
2) prove that a particular AC-OPF problem has no solution;
3) produce a solution that is feasible in the original non-convex problem [7], thus solving the AC-OPF and guaranteeing that the solution is globally optimal.

The ability to provide bounds is particularly important for the numerous mixed-integer nonlinear optimization problems that arise in power system applications. For these reasons, a variety of convex relaxations of the AC-OPF have been developed including, the SDP [11, QC 4, SOC 2, and Convex-DistFlow 3], which are reviewed in detail in this section. Moreover, since the SOC and Convex-DistFlow relaxations have been shown to be equivalent [25], this paper focuses on the SDP, SOC, and QC relaxations only and shows how they are derived from Model 2. The key insight is that each relaxation presents a different approach to convexifying constraints (16b), which are the only source of non-convexity in Model 2.

The Semi-Definite Programming (SDP) Relaxation: exploits the fact that the W variables are defined by $V (V^*)^T$, which ensures that W is positive semi-definite (denoted by $W \geq 0$) and has rank 1 [11, 7, 13]. These conditions are sufficient to enforce constraints (16b), i.e.,

$W_{ij} = V_i V_j^* \ (i,j \in N) \ Leftrightarrow W \geq 0 \land rank(W) = 1$

The SDP relaxation [27, 26] then drops the rank constraint to obtain Model 3.

The Second Order Cone (SOC) Relaxation: convexifies each constraint of (16b) separately, instead of considering them globally as in the SDP relaxation. The SOC relaxation takes
the absolute square of each constraint, refactors it, and then relaxes the equality into an inequality, i.e.,

\[ W_{ij} = V_i V_j^* \quad (19a) \]

\[ W_{ij} W_j^* = V_i V_j^* V_i V_j \quad (19b) \]

\[ |W_{ij}|^2 = W_{ii} W_{jj} \quad (19c) \]

\[ |W_{ij}|^2 \leq W_{ii} W_{jj} \quad (19d) \]

Equation (19b) is a rotated second-order cone constraint which is widely supported by industrial optimization tools. It can, in fact, be rewritten in the standard form of a second-order cone constraint as,

\[ \left( \frac{2W_{ij}}{W_{ii} - W_{jj}} \right) \leq W_{ii} + W_{jj} \quad (20) \]

The complete SOC formulation is presented in Model 4. Note that this relaxation requires fewer \( W \) variables than Model 3. Due to the sparsity of AC power networks, this size reduction can lead to significant memory and computational savings.

The Quadratic Convex (QC) Relaxation: was introduced to preserve stronger links between the voltage variables [3]. It represents the voltages in polar form (i.e., \( V = v \angle \theta \)) and links these real variables to the \( W \) variables, along the lines of [16], [17], [28], [29], using the following equations:

\[ W_{ii} = v_i^2 \quad i \in N \quad (21a) \]

\[ \Re(W_{ij}) = v_i v_j \cos(\theta_i - \theta_j) \quad \forall (i,j) \in E \quad (21b) \]

\[ \Im(W_{ij}) = v_i v_j \sin(\theta_i - \theta_j) \quad \forall (i,j) \in E \quad (21c) \]

The QC relaxation then relaxes these equations by taking tight convex envelopes of their nonlinear terms, exploiting the operational limits for \( v_i, v_j, \theta_i - \theta_j \). The convex envelopes for the square and product of variables are well-known [30], i.e.,

\[ \langle x^2 \rangle_T = \begin{cases} x^2 & \text{T-CONV} \\ x \leq (x_u + x_l) x - x_u x_l & \end{cases} \]

\[ \langle xy \rangle_T = \begin{cases} x y & \text{M-CONV} \\ x y \geq x u y + y u x - x u x l & \end{cases} \]

Under our assumptions that the phase angle bound satisfies \( 0 \leq \theta \Delta \leq \frac{\pi}{2} \) and is symmetric, convex envelopes for sine (S-CONV) and cosine (C-CONV) [4] are given by,

\[ \langle \sin(x) \rangle_S = \begin{cases} \sin(x) & \text{S-CONV} \\ -\sin(x) & \text{S-CONV} \end{cases} \]

\[ \langle \cos(x) \rangle_C = \begin{cases} \cos(x) & \text{C-CONV} \\ -\cos(x) & \text{C-CONV} \end{cases} \]

The complete QC relaxation is presented in Model 5. This model is annotated as C-QC, as the second-order cone constraints use current variables. The motivation for this distinction will become clear in Section V.

### IV. AN ILLUSTRATIVE EXAMPLE

This section illustrates the three main power flow relaxations on the 3-bus network from [31], which has proven to be an excellent test case for power flow relaxations. This system is depicted in Figure 1 and the associated network parameters are given in Table 1. This network is designed to have very few binding constraints. Hence, the generator and line limits are set to large non-binding values, except for the thermal limit constraint on the line between buses 2 and 3, which is set to 50 MVA. In addition to its base configuration, we also consider this network with reduced phase angle difference bounds of 18°. IPOPT [32] is used as a heuristic [33] to find a feasible
solution to the AC-OPF and we measure the optimally gap between the heuristic and a relaxation using the formula

\[
\text{Heuristic} - \text{Relaxation} = \text{Optimality Gap (\%)}
\]

Table I summarizes the results in the base configuration, the SDP relaxation has the smallest optimality gap. In the \(\theta = 18^\circ\) case, the QC relaxation has the smallest optimality gap, while reducing the bound on phase angle differences increases the optimality gap for both the SDP and SOC relaxations. This small network highlights two important results. First, the SDP relaxation does not dominate the QC relaxation and vice versa. Second, the SDP and QC relaxations dominate the SO relaxations. The next two sections prove that this last result holds for all networks.

V. AN ALTERNATE FORM OF THE QC RELAXATION

Section III introduced two types of second-order cone constraints. Model 4 uses a SOC constraint based on the absolute square of the voltage product \(v_i v_j\), i.e.,

\[
|W_{ij}|^2 \leq W_{ii} W_{jj} \tag{25}
\]

while Model 5 uses a SOC constraint based on the absolute square of the power flow \(p_{ii} p_{jj}\), i.e.,

\[
|S_{ij}|^2 \leq W_{ii} l_{ij} \tag{26}
\]

We now show that, in conjunction with the power flow equations \((16a)-(16g)\), these two SOC formulations are equivalent. More precisely, we show that

\[
\begin{align*}
S_{ij} &= Y_{ij}^* W_{ii} - Y_{ij}^* W_{ij} \quad (i,j) \in E \\
S_{ji} &= Y_{ji}^* W_{jj} - Y_{ji}^* W_{ij} \quad (i,j) \in E \\
|W_{ij}|^2 &\leq W_{ii} W_{jj} \quad (i,j) \in E
\end{align*}
\]

(W-SOC)

This equivalence suggests an alternative formulation of the QC relaxation which is given in Model 6 and establishes a clear connection between Models 4 and 5. Throughout this paper, we use \(W\) and \(C\) to denote which of these equivalent formulations is used.

We now prove these results. The following lemma, whose proof is straightforward and can be found in the Appendix, establishes some useful equalities.

Lemma V.1. The following four equalities hold:

1. \(|S_{ij}|^2 = |Y_{ij}|^2 \left(W_{ii}^2 - W_{ii} W_{ij} - W_{ii} W_{ij}^* + |W_{ij}|^2\right)\)
2. \(|W_{ij}|^2 = W_{ii}^2 - W_{ii} Z_{ij}^* S_{ij} - W_{ii} Z_{ij} S_{ij}^* + |Z_{ij}|^2 |S_{ij}|^2\)
3. \(l_{ij} = |Y_{ij}|^2 (W_{ii} + W_{ij} - W_{ii} - W_{ij}^*)\)
4. \(W_{jj} = W_{ii} - Z_{ij}^* S_{ij} - Z_{ij} S_{ij}^* + |Z_{ij}|^2 l_{ij}\)

We are now ready to prove the main result of this section.

Theorem V.2. (C-SOC) is equivalent to (W-SOC).

Proof. The proof is similar in spirit to those presented in [25, 54].

\(W-SOC \Rightarrow C-SOC\) Every solution to (W-SOC) is a solution to (C-SOC). Given a solution to (W-SOC), by equality (3) in Lemma V.1 we assign \(l_{ij}\) as follows:

\[
l_{ij} = |Y_{ij}|^2 (W_{ii} - W_{ij} - W_{ij}^* + W_{jj}) \quad (i,j) \in E
\]

This assignment satisfies the power loss constraint \((22d)\) by definition of the power. It remains to show that second-order cone constraint in (C-SOC) is satisfied. Using equalities (1) and (3) in Lemma V.1 we obtain

\[
\begin{align*}
|S_{ij}|^2 &= |Y_{ij}|^2 \left(W_{ii}^2 - W_{ii} W_{ij} - W_{ii} W_{ij}^* + |W_{ij}|^2\right) \\
|S_{ij}|^2 &\leq |Y_{ij}|^2 \left(W_{ii}^2 - W_{ii} W_{ij} - W_{ii} W_{ij}^* + |W_{ij}|^2\right) \\
|S_{ij}|^2 &\leq |Y_{ij}|^2 \left(W_{ii}^2 - W_{ii} W_{ij} - W_{ii} W_{ij}^* + |W_{ij}|^2\right) \\
|S_{ij}|^2 &\leq |W_{ii}| |Y_{ij}|^2 \left(W_{ii} - W_{ij} - W_{ij}^* + W_{jj}\right) \\
|S_{ij}|^2 &\leq |W_{ii}| |Y_{ij}|^2 \left(W_{ii} - W_{ij} - W_{ij}^* + W_{jj}\right)
\end{align*}
\]

(C-SOC) \(\Rightarrow W-SOC\) Every solution to (C-SOC) is a solution to (W-SOC). We show that the values of \(W_{ij}\) in (C-SOC) satisfy the second-order cone constraint in (W-SOC).
Using equalities (2) and (4) in Lemma V.1 and the fact that $W_{ii} = W_{ii}^*$ since $W_{ii}$ is a real number, we have

$$|W_{ij}|^2 = W_{ii}^2 - W_{ij}^* Z_{ij}^* S_{ij} - W_{ii} Z_{ij} S_{ij}^* + |Z_{ij}^2 S_{ij}|^2$$

$$|W_{ij}|^2 \leq W_{ii}^2 - W_{ij}^* Z_{ij}^* S_{ij} - W_{ii} Z_{ij} S_{ij}^* + |Z_{ij}^2 W_{ii} S_{ij}|$$

$$|W_{ij}|^2 \leq W_{ii} (W_{ij} - Z_{ij}^* S_{ij} - Z_{ij} S_{ij}^* + |Z_{ij}^2 S_{ij}|)$$

$$|W_{ij}|^2 \leq W_{ii} W_{jj}$$

and the result follows.

**Corollary V.3.** Model [5] is equivalent to Model [6].

Computational results on these two formulations are presented in the Appendix. The main message is that the C-SOC formulation is preferable to W-SOC in the current state of the solving technology, especially on very large networks.

It is important to note that, for clarity, the proofs are presented on the purest version of the AC power flow equations. Transmission system test cases typically include additional parameters such as bus shunts, line charging, and transformers. Proofs that these results can be extended to include the additional parameters in transmission system test cases are presented in the Appendix.

**VI. RELATIONS OF THE POWER FLOW RELAXATIONS**

We are now in a position to state the relationships between the convex relaxations. Recall that model $M_1$ is a relaxation of model $M_2$, denoted by $M_2 \subseteq M_1$, if the solution set of $M_2$ is included in the solution set $M_1$. We use $M_1 \neq M_2$ to denote the fact that neither $M_2 \subseteq M_1$ nor $M_1 \subseteq M_2$ holds. Since our relaxations have different sets of variables, we define the solution set as the assignments to the $W_{ij}$ variables.

**Theorem VI.1.** The following properties, illustrated in Figure 2 hold:

1) $SDP \subseteq SOC$.
2) $SDP \neq QC$.
3) $QC \subseteq SOC$.

**Proof.** Properties (1) and (2) follows from [18] and Section IV respectively. For Property (3), observe that the set of constraints in Model [6] (W-QC) is a superset of those in Model [4]. The result follows from Corollary V.3.

Observe that the additional constraints [22a]–[22c] in the QC formulations are parameterized by the $\theta$. As $\theta$ grows larger, the QC model reduces to the SOC model. Clearly, the strength of the QC relaxation is sensitive to this input parameter, as illustrated in Section IV.

**VII. COMPUTATIONAL EVALUATION**

This section presents a computational evaluation of the relaxations and address the following questions:

1) How big are the optimality gaps in practice?
2) What are the runtime requirements of the relaxations?
3) How robust is the solving technology for the relaxations?

The relaxations were compared on 105 state-of-the-art AC-OPF transmission system test cases from the NESTA v0.4.0 archive [35]. These test cases range from as few as 3 buses to as many as 9000 and consist of 35 different networks under a typical operating condition (TYP), a congested operating condition (API), and a small angle difference condition (SAD). A small angle difference condition (SAD).

**Experimental Setting:** All of the computations are conducted on Dell PowerEdge R415 servers with Dual 2.8GHz AMD 6-Core Opteron 4184 CPUs and 64GB of memory. IPOPT 3.12 [32] with linear solver ma27 [36], as suggested by [37], was used as a heuristic for finding locally optimal feasible solutions to the non-convex AC-OPF formulated in AMPL [38]. The SDP relaxation was executed on the state-of-the-art implementation [39] which uses a branch decomposition [40] with a minor extension to add constraint (16i). The SDP solver SDPT3 4.0 [41] was used with the modifications suggested in [39]. The second-order cone models were formulated in AMPL and IPOPT was used to solve the models. Numerical stability appears to be a significant challenge on the power networks with more than 1000 buses [42]. Note that IPOPT is single-threaded and does not take advantage of the multiple cores available in the computation servers. This gives some computational advantage to the SDP solver, which utilizes multiple cores.

**Challenging Test Cases:** We observe that 52 of the 105 test cases considered have an optimality gap of less than 1.0% with the SOC relaxation. Such test cases are not particularly useful for this study as the improvements of the SDP and QC models are minor. Hence, we focus our attention on the 53 test cases where the SOC optimality gap is greater than 1.0%. The results are displayed in Table III.

**A. The Quality of the Relaxations**

The first six columns of Table III present the optimality gaps for each of the relaxations on the 53 challenging NESTA test cases. Note that bold values indicate cases where the relaxation produced a solution to the non-convex AC power flow problem. The table indicates that this is a rare occurrence in the cases considered.

**The SDP Relaxation:** Overall, the SDP relaxation tends to be the tightest, often featuring optimality gaps below 1.0%. In 5 of the 53 cases, the SDP relaxation even produces a feasible AC power flow solution (as first observed in [2]). However, with six notable cases where the gap is above 5%, it is clear that small gaps are not guaranteed. In some cases, the optimality gap can be as large as 30%.

*Nine test cases based on the EIR Grid network were omitted from evaluation because the AC-OPF-W-SDP solver did not support inactive buses.*
A significant issue with the SDP relaxation is the reliability of the solving technology. Even after applying the solver modifications suggested in [39], the solver fails to converge to a solution before hitting the default iteration limit on 8 of the 53 test cases shown, it reports numerical accuracy warnings. For these cases, the solver reports that it has reached the iteration limit without converging to a solution. The QC and SOC Relaxations: As suggested by the theoretical study in Section VI, the QC relaxation is quite similar to the SOC relaxation. However, when the phase angle difference bounds are tight (e.g., in the SAD cases), the QC relaxation is significantly better than the SOC relaxation. On average, the SDP relaxation...
dominates the QC and SOC relaxations. However, there are several notable cases (e.g. nesta_case24_ieee_rts_sad, nesta_case29_edin_sad, nesta_case73_ieee_rts_sad) where the QC relaxation dominates the SDP relaxation.

The Copper Plate (CP) Relaxation: This relaxation indicates the cost of supplying power to the loads when there are no line losses or network constraints [43], and is included in the table as a point of reference. Note that this relaxation cannot be applied to networks containing lines with negative resistance or impedance, as indicated by “n.a.”.

B. The Performance of the Relaxations

Detailed runtime results for the heuristic solution method and the relaxations are presented in the last four columns of Table III. The AC heuristic is fast, often taking less than 1 second on test cases with less than 1000 buses. The SOC relaxation most often has very similar performance to the AC heuristic. The additional constraints in the QC relaxation add a factor 2–5 on top of the SOC relaxation. In contrast to these other methods, the SDP relaxation stands out, taking 10–100 times longer. It is interesting to observe, in the 5 cases where the SDP relaxation finds an AC-feasible solution, the heuristic finds a solution of equal quality in a fraction of the time. Focusing on the test cases where the SDP fails to converge, we observe that the failure occurs after several minutes of computation, further emphasizing the reliability issue.

VIII. Conclusion

This paper compared the QC relaxation of the power flow equations with the well-understood SDP and SOC relaxations both theoretically and experimentally. Its two main contributions are as follows:

1) The QC relaxation is stronger than the SOC relaxation and neither dominates nor is dominated by the SDP relaxation.
2) Computational results on optimal power flow show that the QC relaxation may bring significant benefits in accuracy over the SOC relaxation, especially for tight bounds on phase angle differences, for a reasonable loss in efficiency. In addition, they show that, with existing solvers, the SOC and QC relaxations are significantly faster and more reliable than the SDP relaxation.

There are two natural frontiers for future work on these relaxations: One is to utilize these relaxations in power system applications that are modeled as mixed-integer nonlinear optimization problems, such as the Optimal Transmission Switching, Unit Commitment, or Transmission Network Expansion Planning. Indeed, Mixed-Integer Quadratic Programming solvers are already being used to extend these relaxations to richer power system applications [41, 44, 45, 46, 47]. The other frontier is to develop novel methods for closing the significant optimality gaps that remain on a variety of test cases considered here.

Acknowledgements

The authors would like to thank the four anonymous reviewers for their insightful suggestions for improving this work.

NICTA is funded by the Australian Government through the Department of Communications and the Australian Research Council through the ICT Centre of Excellence Program.

References


Carleton Coffrin received a B.Sc. in Computer Science and a B.F.A. in Theatrical Design from the University of Connecticut, Storrs, CT and a M.S. and Ph.D. from Brown University, Providence, RI. He is currently a staff researcher at National ICT Australia where he studies the application of optimization methods to problems in power systems.

Hassan L. Hijazi received a Ph.D. in Computer Science from AIX-Marseille University while working at Orange Labs - France Telecom R&D from 2007 to 2010. He then joined the Optimization Group at the Computer Science Laboratory of the Ecole Polytechnique-France where he stayed until late 2012. He is currently a senior research scientist at National ICT Australia and a senior lecturer at the Australian National University. His main field of research is mixed-integer nonlinear optimization and applications in network-based problems, where he has given contributions both in theory and practice.

Pascal Van Hentenryck received the undergraduate and Ph.D. degrees from the University of Namur, Namur, Belgium. He currently leads the Optimization Research Group at National ICT Australia and holds the Vice-Chancellor Chair in data-intensive computing at the Australian National University, Canberra, Australia. Prior to that, he was a Professor with Brown University, Providence, RI, USA. His current research interests are in optimization with applications to disaster management, power systems, and transportation.
Is derived using the following steps:

\[ S_{ij} = Y_{ij}^* W_{ii} - Y_{ij}^* W_{ij} \]

\[ |S_{ij}|^2 = |Y_{ij}|^2 (W_{ii}^2 - W_{ii} W_{ij} - W_{ii} W_{ij}^* + |W_{ij}|^2) \]

Property 3 – the absolute square of current –

\[ l_{ij} = |Y_{ij}|^2 (W_{ii} + W_{ij} - W_{ij}^*) \]

is derived using the following steps:

\[ I_{ij} = Y_{ij} (V_i - V_j) \]

\[ |I_{ij}|^2 = |Y_{ij}|^2 (V_i^* - V_i V_j^* - V_i^* V_j + V_j^* V_j) \]

\[ l_{ij} = |Y_{ij}|^2 (W_{ii} - W_{ij} - W_{ij}^* + W_{jj}) \]

Property 4 – voltage drop –

\[ W_{jj} = W_{ii} - Z_{ij}^* S_{ij} - Z_{ij} S_{ij}^* + |Z_{ij}|^2 l_{ij} \]

is derived using the following steps:

\[ W_{jj} = W_{ij} \]

\[ W_{jj} = W_{ii} - W_{ii} + W_{ij} - W_{ii} + W_{ij}^* + W_{ii} - W_{ij} - W_{ij}^* + W_{jj} \]

\[ W_{jj} = W_{ii} - W_{ij}^* + W_{jj} + W_{ii} - W_{ij}^* + W_{jj} + |Z_{ij}|^2 l_{ij} \]

\[ W_{jj} = W_{ii} - Z_{ij} S_{ij} - Z_{ij} S_{ij}^* + |Z_{ij}|^2 l_{ij} \]

Note that property 3 is used in the second step.

---

**Model 7 AC-OPF-W with Extensions**

**variables:** \( S_i^q (\forall i \in N), W_{ij}(\forall i, j \in N) \)

**minimize:** (16a)

**subject to:** (16b), (16d), (16b), (16d)

\[ S_i^q - S_i^q - Y_{ij}^* W_{ii} = \sum_{(i,j) \in E \cup E^T} S_{ij} \forall i \in N \] (34b)

\[ S_{ij} = \left( Y_{ij}^* \frac{i b_{ij}^c}{2} \right) \left( \frac{W_{ii}}{|T_{ij}|^2} - Y_{ij}^* \frac{W_{ij}}{|T_{ij}|^2} \right) \forall i, j \in E \] (34c)

\[ S_{ji} = \left( Y_{ij}^* \frac{i b_{ij}^c}{2} \right) \left( W_{jj} - Y_{ij}^* \frac{W_{ij}}{|T_{ij}|^2} \right) \forall i, j \in E \] (34d)

---

**Extensions for Transmission System Test Cases**

In the interest of clarity, properties of AC Power Flow, and their relaxations, are most often presented on the purest version of the AC power flow equations. However, transmission system test cases include additional parameters such as bus shunts, line charging, and transformers, which complicate the AC power flow equations significantly. Model 7 presents the AC Optimal Power Flow problem (similar to Model 2) with these extensions. In the rest of this section, we show that the results of Section V continue to hold in this extended power flow model.

**The Two SOC Formulations:** In this extended power flow formulation, the second-order cone constraint based on the absolute square of the voltage product (2) remains the same, i.e.,

\[ |W_{ij}|^2 \leq W_{ii} W_{jj} \] (35)

However, the constraint based on the absolute square of the power flow (3) is updated to include the transformer tap ratio as follows:

\[ |S_{ij}|^2 \leq \frac{W_{ii}}{|T_{ij}|^2} l_{ij} \] (36)

and the power loss constraint (22d) is updated to

\[ S_{ij} + S_{ji} = Z_{ij} l_{ij} + \left( \frac{b_{ij}^e}{2} \right)^2 \frac{W_{ii}}{|T_{ij}|^2} + b_{ij}^e S_{ij} + \frac{b_{ij}^c}{2} \left( \frac{W_{ii}}{|T_{ij}|^2} + W_{jj} \right) \] (37)

We now show that, when the power flow equations (34c)–(34d) are present in the model, these two versions of the second-order cone constraints are also equivalent, i.e.,

\[ S_{ij} = \left( Y_{ij}^* \frac{i b_{ij}^c}{2} \right) \left( \frac{W_{ii}}{|T_{ij}|^2} - Y_{ij}^* \frac{W_{ij}}{|T_{ij}|^2} \right) \forall i, j \in E \]

\[ S_{ji} = \left( Y_{ij}^* \frac{i b_{ij}^c}{2} \right) \left( W_{jj} - Y_{ij}^* \frac{W_{ij}}{|T_{ij}|^2} \right) \forall i, j \in E \]

\[ |W_{ij}|^2 \leq W_{ii} W_{jj} \] (W-E-SOC)
is equivalent to

\[
S_{ij} = \left( Y_{ij}^* - i \frac{b_c^e}{2} \right) \frac{W_{ii}}{|T_{ij}|^2} - Y_{ij}^* \frac{W_{ij}}{|T_{ij}|^2} \quad (i, j) \in E
\]
\[
S_{ji} = \left( Y_{ji}^* - i \frac{b_c^e}{2} \right) W_{jj} - Y_{ji}^* \frac{W_{jj}}{|T_{ij}|^2} \quad (i, j) \in E
\]
\[
S_{ij} + S_{ji} = Z_{ij} \left( i_{ij} + \frac{(b_c^e)}{2} \right) \frac{W_{ii}}{|T_{ij}|^2} + b_c^e \Im(S_{ij})
\]
\[
- i \frac{b_c^e}{2} \left( \frac{W_{ii}}{|T_{ij}|^2} + W_{jj} \right) \quad (i, j) \in E
\]
\[
|S_{ij}|^2 \leq \frac{|S_{ij}|^2}{|T_{ij}|^2} + I_{ij} \quad (i, j) \in E.
\]

(C-E-SOC)

We begin by redeveloping the properties of Lemma V.1 in the extended model.

The Equalities: As both models contain constraints (33a–34d), the properties arising from these equations can be transferred between both models.

Proof.

Property 1 – the absolute square of power –

\[
|S_{ij}|^2 = |Y_{ij}|^2 \left( \frac{W_{ii}^2}{|T_{ij}|^2} - \frac{W_{ii} W_{ij}}{|T_{ij}|^2} - \frac{W_{ii} W_{ij}}{|T_{ij}|^2} \frac{W_{ij}}{|T_{ij}|^2} + \frac{|W_{ij}|^2}{|T_{ij}|^2} \right)
\]
\[
- \left( \frac{b_c^e}{2} \right)^2 \frac{W_{ii}^2}{|T_{ij}|^2} - b_c^e \frac{W_{ii}}{|T_{ij}|^2} \Im(S_{ij})
\]

(38)

The derivation follows similarly to the one presented earlier and the details are left to the reader. The only delicate point is to observe that three separate terms in the initial expansion can be collected into \( \Im(S_{ij}) \).

Property 2 – the absolute square of the voltage product –

\[
|W_{ij}|^2 = (1 - b_c^e \Im(Z_{ij})) \frac{W_{ii}^2}{|T_{ij}|^2} - W_{ii} W_{ij} S_{ij} - W_{ii} Z_{ij} S_{ij}
\]
\[
+ |Z_{ij}|^2 \left( |T_{ij}|^2 |S_{ij}|^2 \right) + \left( \frac{b_c^e}{2} \right)^2 \frac{W_{ii}^2}{|T_{ij}|^2} + W_{ii} b_c^e \Im(S_{ij})
\]

(39)

The derivation follows similarly to the one presented earlier and the details are left to the reader.

Property 3 – the absolute square of current –

\[
l_{ij} = |Y_{ij}|^2 \left( \frac{W_{ii}}{|T_{ij}|^2} - \frac{W_{ii} W_{ij}}{|T_{ij}|^2} - \frac{W_{ii}}{|T_{ij}|^2} + W_{jj} \right)
\]
\[
- \left( \frac{b_c^e}{2} \right)^2 \frac{W_{ii}}{|T_{ij}|^2} - b_c^e \Im(S_{ij})
\]

(40)

After observing that the extension of Ohm’s Law in this model is given by

\[
I_{ij} = \left( Y_{ij} + i \frac{b_c^e}{2} \right) \frac{V_i}{T_{ij}} - Y_{ij} V_j \quad (i, j) \in E
\]

the derivation follows similarly to the one presented earlier and the details are left to the reader.

Property 4 – voltage drop –

\[
W_{jj} = (1 - b_c^e \Im(Z_{ij})) \frac{W_{ii}}{|T_{ij}|^2} - Z_{ii} S_{ij} - Z_{ij} S_{ij}
\]
\[
+ |Z_{ij}|^2 \left( \frac{(b_c^e)^2}{2} \frac{W_{ii}}{|T_{ij}|^2} + b_c^e \Im(S_{ij}) \right)
\]

(42)

The proof follows similarly to the one presented earlier and the details are left to the reader.

With these core properties updated, we are now ready to extend the proof from Section V.

Theorem A.1. (C-E-SOC) is equivalent to (W-E-SOC).

Proof. The proof follows the one presented in Section V.  

1) C-E-SOC \( \Rightarrow \) W-E-SOC Every solution to (C-E-SOC) is a solution to (W-E-SOC). Given a solution to (W-SOC), by equality (3), we assign \( l_{ij} \) as follows:

\[
l_{ij} = |Y_{ij}|^2 \left( \frac{W_{ii}}{|T_{ij}|^2} - \frac{W_{ij}}{|T_{ij}|^2} - \frac{W_{ii}^*}{|T_{ij}|^2} + W_{jj} \right)
\]
\[
- \left( \frac{b_c^e}{2} \right)^2 \frac{W_{ii}}{|T_{ij}|^2} - b_c^e \Im(S_{ij}) \quad (i, j) \in E
\]

(43)

This assignment satisfies the power loss constraint (37) by definition of the power. It remains to show that second-order cone constraint in (C-E-SOC) is satisfied. Using equalities (1) and (3), we obtain

\[
|S_{ij}|^2 = |Y_{ij}|^2 \left( \frac{W_{ii}^2}{|T_{ij}|^2} - \frac{W_{ii} W_{ij}}{|T_{ij}|^2} - \frac{W_{ii} W_{ij}}{|T_{ij}|^2} \frac{W_{ij}}{|T_{ij}|^2} + \frac{|W_{ij}|^2}{|T_{ij}|^2} \right)
\]
\[
- \left( \frac{b_c^e}{2} \right)^2 \frac{W_{ii}^2}{|T_{ij}|^2} - b_c^e \frac{W_{ii}}{|T_{ij}|^2} \Im(S_{ij})
\]

(44a)

\[
|S_{ij}|^2 \leq \frac{|Y_{ij}|^2}{|T_{ij}|^2} \left( \frac{W_{ii}^2}{|T_{ij}|^2} - \frac{W_{ii} W_{ij}}{|T_{ij}|^2} - \frac{W_{ii} W_{ij}}{|T_{ij}|^2} \frac{W_{ij}}{|T_{ij}|^2} + \frac{|W_{ij}|^2}{|T_{ij}|^2} \right)
\]
\[
- \left( \frac{b_c^e}{2} \right)^2 \frac{W_{ii}^2}{|T_{ij}|^2} - b_c^e \frac{W_{ii}}{|T_{ij}|^2} \Im(S_{ij})
\]

(44b)

\[
|S_{ij}|^2 \leq \frac{W_{ii}}{|T_{ij}|^2} \left( |Y_{ij}|^2 \left( \frac{W_{ii}^2}{|T_{ij}|^2} - \frac{W_{ii} W_{ij}}{|T_{ij}|^2} - \frac{W_{ii} W_{ij}}{|T_{ij}|^2} \frac{W_{ij}}{|T_{ij}|^2} + \frac{|W_{ij}|^2}{|T_{ij}|^2} \right) \right)
\]
\[
- \left( \frac{b_c^e}{2} \right)^2 \frac{W_{ii}^2}{|T_{ij}|^2} - b_c^e \frac{W_{ii}}{|T_{ij}|^2} \Im(S_{ij})
\]

(44c)

\[
|S_{ij}|^2 \leq \frac{W_{ii}}{|T_{ij}|^2} l_{ij}
\]

(44d)

b) C-E-SOC \( \Rightarrow \) W-E-SOC Every solution to (C-E-SOC) is a solution to (W-E-SOC). We show that the values of \( W_{ij} \) in (C-E-SOC) satisfy the second-order cone constraint
\( W_{ij} \) in (W-E-SOC). Using equalities (2) and (4) and the fact that \( W_{ii} = W_{ii}^* \) since \( W_{ii} \) is a real number, we have

\[
|W_{ij}|^2 = \left(1 - b_{ij}^c \Im(Z_{ij})\right) \frac{W_{ii}}{|T_{ij}|^2} - W_{ii}^* Z_{ij} S_{ij} - W_{ii} Z_{ij} S_{ij}^* + |Z_{ij}|^2 \left(\frac{|T_{ij}|^2|S_{ij}|^2 + \left(\frac{b_{ij}^c}{2}\right)^2 \frac{W_{ii}^2}{|T_{ij}|^2} + W_{ii} b_{ij}^c \Im(S_{ij})}\right) \tag{45a}
\]

\[
|W_{ij}|^2 \leq \left(1 - b_{ij}^c \Im(Z_{ij})\right) \frac{W_{ii}}{|T_{ij}|^2} - W_{ii}^* Z_{ij} S_{ij} - W_{ii} Z_{ij} S_{ij}^* + |Z_{ij}|^2 \left(W_{ii} l_{ij} + \left(\frac{b_{ij}^c}{2}\right)^2 \frac{W_{ii}}{|T_{ij}|^2} + W_{ii} b_{ij}^c \Im(S_{ij})\right) \tag{45b}
\]

\[
|W_{ij}|^2 \leq W_{ii} \left(1 - b_{ij}^c \Im(Z_{ij})\right) \frac{W_{ii}}{|T_{ij}|^2} - Z_{ij}^* S_{ij} - Z_{ij} S_{ij}^* + W_{ii} \left(\frac{|Z_{ij}|^2}{2} \frac{W_{ii}}{|T_{ij}|^2} + b_{ij}^c \Im(S_{ij})\right) \tag{45c}
\]

\[
|W_{ij}|^2 \leq W_{ii} W_{jj} \tag{45d}
\]

and the result follows.

Corollary A.2. Model 5 is equivalent to Model 6 in the extended AC Power Flow formulation from Model 7.

Comparison of SOC formulations

Section V proposed two equivalent formulations of the second-order cone constraints for power flow relaxations. Although both formulations define the same convex set, it is unclear if they have the same performance characteristics. For example, the current-based constraint (C-SOC) has more constraints and more variables than the voltage-product constraint (W-SOC). All other aspects being equal, one would expect (C-SOC) to be slower than (W-SOC). This section investigates the performance implications of these two formulations on both the QC and SOC power flow relaxations. Four power flow relaxations are considered, W-SOC (Model 4), C-SOC (Model 4 with (C-SOC)), W-QC (Model 5 with (W-SOC)), and C-QC (Model 5). To test the performance of these relaxations, each model is evaluated on 105 state-of-the-art AC-OPF transmission system test cases from the NESTA v0.4.0 archive [35]. Figure 3 compares the two variants of the SOC relaxation and Figure 4 compares two variants on the QC relaxation.

Both figures indicate that the two formulations are very similar for small test cases but, on the larger test cases (i.e., with more than 1000 buses), the C-QC formulation has a faster convergence rate, in IPOPT. This suggests that, despite its increased size, the C-QC formulation originally presented in [4] is preferable from a performance standpoint and that the C-SOC formulation may be preferable on very large networks (e.g., above 9000 buses).