A Linear-Programming Approximation of AC Power Flows

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August 19, 2014

Abstract

Linear active-power-only power flow approximations are pervasive in the planning and control of power systems. However, AC power systems are governed by a system of nonlinear non-convex power flow equations. Existing linear approximations fail to capture key power flow variables including reactive power and voltage magnitudes, both of which are necessary in many applications that require voltage management and AC power flow feasibility. This paper proposes novel linear-programming models (the LPAC models) that incorporate reactive power and voltage magnitudes in a linear power flow approximation. The LPAC models are built on a polyhedral relaxation of the cosine terms in the AC equations, as well as Taylor approximations of the remaining nonlinear terms. Experimental comparisons with AC solutions on a variety of standard IEEE and MATPOWER benchmarks show that the LPAC models produce accurate values for active and reactive power, phase angles, and voltage magnitudes. The potential benefits of the LPAC models are illustrated on two “proof-of-concept” studies in power restoration and capacitor placement.

Nomenclature

\( \tilde{V} = v + i\theta \) - AC voltage
\( \tilde{S} = p + iq \) - AC power
\( \tilde{Y} = g + ib \) - Line admittance
\( \tilde{V} = |\tilde{V}| e^{i\theta} \) - Polar form
\( |\tilde{V}| \) - Hot-start voltage magnitude
\( |\tilde{V}| \) - Target voltage magnitude
\( \phi \) - Voltage magnitude change

\( \mathcal{PN} \) - Power network
\( \mathcal{P} \) - Set of buses in a power network
\( \mathcal{L} \) - Set of lines in a power network
\( \mathcal{G} \) - Set of generator buses
\( s \) - Slack bus
\( \hat{x} \) - Approximation of \( x \)
\( \bar{x} \) - Upper bound of \( x \)
\( \underline{x} \) - Lower bound of \( x \)

1 Introduction

Optimization technology is widely used in modern power systems [36] and has resulted in millions of dollars in savings annually [38]. But the increasing role of demand response, the integration of renewable
sources of energy, and the desire for more automation in fault detection and recovery pose new challenges for the planning and control of electrical power systems [32]. Power grids now need to operate in more stochastic environments and under varying operating conditions, while still ensuring system reliability and security.

Optimization of power systems encompasses a broad spectrum of problem domains, including optimal power flow [59, 3, 55, 26, 17, 50, 11, 2, 23, 34, 35], LMP-base market calculations [39, 48, 37], transmission switching [14, 18, 22], real-time security-constrained dispatch [60, 53], day-ahead security-constrained unit commitment [27, 49, 41], distribution network configuration [7, 44], capacitor placement [1, 12, 19], expansion planning [4, 31, 20, 28, 25, 21, 58, 13, 54], vulnerability analysis [46, 47, 40, 45, 5], and power system restoration [8, 57] to name a few. Some of these use active power only, while others consider both active and reactive power.

Restricting attention to active power is often appealing computationally as the nonlinear AC power flow equations can then be approximated by a set of linear equations that define the so-called DC power flow model (DC). Under normal operating conditions and with some adjustment for line losses, the DC model produces a reasonably accurate approximation of the AC power flow equations for active power [52]. Moreover, the DC model can be embedded in Mixed-Integer Programming (MIP) models for a variety of optimization applications in power system operations. This is particularly attractive as the computational efficiency of MIP solvers has significantly improved over the last two decades [6].

However, the DC model does not capture reactive power and hence cannot be used for applications such as capacitor placement and voltage management to name only two. Moreover, the accuracy of the DC model outside normal operating conditions is an open point of discussion [39, 52, 43, 10, 9]. This in turn raises concerns for other applications such as transmission planning, vulnerability analysis, and power restoration, which may return infeasible or suboptimal solutions when the DC model is used to approximate the AC power flow equations. As a result, these applications often turn to nonlinear programming techniques [17, 34, 35, 37], iterative heuristics and decomposition [2, 23, 21, 40], model relaxation [26, 1], tabu search [19], and genetic algorithms [12, 28] to ensure feasibility. These techniques often require extensive tuning for each problem domain, may consume significant computational resources, and cannot guarantee global optimality.

This paper aims to narrow the gap between the DC and AC power flow equations. It presents, for the first time, linear programs that approximate the AC power flow equations and capture reactive power and voltage magnitudes accurately. These linear programs, called the LPAC models, are based on three key ideas:

1. Both the voltage phase angle and magnitude are important for modeling active and reactive power;
2. A nice polyhedral relaxation of the cosine exits in context of power system design and operation;
3. The nominal operating point of the power system allows a Taylor series to reasonably approximate the remaining nonlinear terms.

These LPAC models deliver the computational benefits of the DC approximation (i.e. industrial strength MIP solvers) to problem domains where the DC approximation is either too inaccurate or inapplicable, due to reactive power or voltage requirements. The LPAC models have been evaluated experimentally over a number of standard benchmarks under normal operating conditions and various contingencies. Experimental comparisons with AC solutions on standard IEEE and MATPOWER benchmarks show that the LPAC models are highly accurate for active and reactive power, phase angles, and voltage magnitudes. Moreover, the LPAC models can be integrated in MIP models for applications reasoning about reactive power (e.g., capacitor placement) or topological changes (e.g., transmission planning, vulnerability analysis, and power restoration). They thus provide a valuable alternative to heuristic and local methods traditionally used for these applications.

The rest of this paper presents a rigorous and systematic derivation of the LPAC models, experimental results about their accuracy, and applications to power restoration and capacitor placement. Section 2 reviews the AC power flow equations. Section 3 derives the LPAC models and Section 4 presents the experimental results on thier accuracy. Section 5 presents the “proof-of-concept” experiments in power restoration and capacitor placement to demonstrate potential applications of the LPAC models. Section 6 discusses related work and Section 7 concludes the paper.
2 Review of AC Power Flow

A power network, \( \mathcal{P}N \), is composed of several types of components such as buses, lines, generators, and loads. The network can be interpreted as a graph where the set of buses \( N \) are the nodes and the set of lines \( L \) are the edges. For any AC power flow study, we also need the set of generators \( G \subseteq N \) and the slack bus \( s \in G \).\(^1\) Every bus \( n \in N \) in the network has two properties, a voltage \( \tilde{V}_n = v_n + j\theta_n \) and a power \( \tilde{S}_n = p_n + iq_n \), both of which are complex numbers. Each line \( \langle n, m \rangle \in L \) has an admittance \( \tilde{Y}_{nm} = g_{nm} + j\,b_{nm} \), also a complex number. These network values are connected by two fundamental physical laws, Kirchhoff’s Current Law, 

\[
\tilde{S}_n = \sum_{\langle n,m \rangle \in L} \tilde{S}_{nm} \quad \forall \, n \in N \tag{1}
\]

and Ohm’s Law,

\[
\tilde{S}_{nm} = \tilde{V}_n \tilde{V}_m^* \tilde{Y}_{nm} - \tilde{V}_m \tilde{V}_n^* \tilde{Y}_{nm}^* \quad \forall \, \langle n,m \rangle \in L \tag{2}
\]

It is important to notice that the line power flows are not symmetric (i.e., \( \tilde{S}_{nm} \neq \tilde{S}_{mn} \) in general). Due to this asymmetry, it is essential that \( L \) contains both \( \langle n,m \rangle \) and \( \langle m,n \rangle \) for each line in the network.

One additional side constraint is critical to modeling AC power flows. Generators in the power system are the voltage-controlled physical laws, Kirchhoff’s Current Law, 

\[
p_n = \sum_{\langle n,m \rangle \in L} p_{nm} \quad \forall \, n \in N \tag{3}
\]

\[
q_n = \sum_{\langle n,m \rangle \in L} q_{nm} \quad \forall \, n \in N \tag{4}
\]

\[
p_{nm} = |\tilde{V}_n|^2 \, g_{nm} - |\tilde{V}_n||\tilde{V}_m| g_{nm} \cos(\theta_n - \theta_m) - |\tilde{V}_n||\tilde{V}_m| b_{nm} \sin(\theta_n - \theta_m) \quad \forall \, \langle n,m \rangle \in L \tag{5}
\]

\[
q_{nm} = -|\tilde{V}_n|^2 b_{nm} + |\tilde{V}_n||\tilde{V}_m| b_{nm} \cos(\theta_n - \theta_m) - |\tilde{V}_n||\tilde{V}_m| g_{nm} \sin(\theta_n - \theta_m) \quad \forall \, \langle n,m \rangle \in L \tag{6}
\]

This work adopts this expansion of the AC power flow equations. As the derivation in Section 3 will demonstrate, the polar representation of voltage is useful for building approximations around the nominal operating point.

2.1 The Traditional Real Number Representation

To model the AC power flow, we need to expand the complex representation into real numbers. Complex numbers may be represented in rectangular or polar coordinates, i.e., \( \tilde{V} = v + j\theta = |\tilde{V}|\angle\theta \). Because power networks operate near a nominal voltage (i.e., \( \tilde{V}_n \approx 1.0 \angle 0 \)), it is common to represent the voltages in polar coordinates and the remaining terms (power and admittance) in rectangular coordinates. Expanding equations (1) and (2) in this way yields

\[
p_n = \sum_{\langle n,m \rangle \in L} p_{nm} \quad \forall \, n \in N \tag{3}
\]

\[
q_n = \sum_{\langle n,m \rangle \in L} q_{nm} \quad \forall \, n \in N \tag{4}
\]

\[
p_{nm} = |\tilde{V}_n|^2 g_{nm} - |\tilde{V}_n||\tilde{V}_m| g_{nm} \cos(\theta_n - \theta_m) - |\tilde{V}_n||\tilde{V}_m| b_{nm} \sin(\theta_n - \theta_m) \quad \forall \, \langle n,m \rangle \in L \tag{5}
\]

\[
q_{nm} = -|\tilde{V}_n|^2 b_{nm} + |\tilde{V}_n||\tilde{V}_m| b_{nm} \cos(\theta_n - \theta_m) - |\tilde{V}_n||\tilde{V}_m| g_{nm} \sin(\theta_n - \theta_m) \quad \forall \, \langle n,m \rangle \in L \tag{6}
\]

This work adopts this expansion of the AC power flow equations. As the derivation in Section 3 will demonstrate, the polar representation of voltage is useful for building approximations around the nominal operating point.

2.2 The DC Power Flow Model

Many variants of the DC power flow model exist [24, 61, 42, 15]. An in-depth discussion can be found in [52]. For brevity, we only review the simplest and most popular variant of the DC model, which is derived from the AC equations through a series of approximations justified by the design and operation of the network. In particular, the DC model assumes that (1) the susceptance is large relative to the conductance \( |g| \ll |b| \); (2) the phase angle difference is small enough to ensure \( \sin(\theta_n^0 - \theta_m^0) \approx \theta_n^0 - \theta_m^0 \) and \( \cos(\theta_n^0 - \theta_m^0) \approx 1.0 \); and (3) the voltage magnitudes \( |\tilde{V}| \) are close to 1.0 and do not vary significantly. Under these assumptions, equations (5) and (6) reduce to

\[
p_{nm} = -b_{nm} (\theta_n^0 - \theta_m^0) \tag{7}
\]

This simple linear formulation has been used in many frameworks for decision support in power systems [39, 14, 4, 47, 8, 57] and is used as the baseline in the experimental results of this paper.

\(^1\)The slack bus is an arbitrary generator bus that is used as a reference for ensuring consistency among different calculations and studies.
3 Linear-Programming Approximations

This section presents the linear-programming approximations of the AC power flow equations. Their goal is to extend the modeling power of the DC model to capture voltage magnitudes and reactive power, while still retaining the attractive computational properties of the DC model.

3.1 Hot-Start and Cold-Start Contexts

To understand the approximations, it is important to distinguish between hot-start and cold-start application contexts [52]. In hot-start contexts, a solved AC base-point solution (i.e., a solution to the AC equations) is available and hence the model has at its disposal reasonable voltage magnitudes. In cold-start contexts, no such solved AC base-point solution is available and it can be “maddeningly difficult” [39] to obtain one by simulation of the network. Hot-start models are suitable for applications in which the network topology is relatively stable, e.g., in LMP-base market calculations, optimal line switching, distribution configuration, and real-time security constrained economic dispatch. Cold-start models are used when no operational network or AC solution is available, e.g., in long-term planning studies.

We also introduce the concept of warm-start context, in which the model has at its disposal target voltage magnitudes (e.g., voltage magnitudes from normal operating conditions) but an actual solution may not exist for these targets. Warm-start models are particularly useful for power restoration applications in which the goal is to return to normal operating conditions as quickly as possible.

The remainder of this section presents the hot-start, warm-start, and cold-start LPAC models in stepwise refinements. It also discusses how models can be generalized to include generation and load shedding, remove the slack bus, impose constraints on voltages and reactive power, and capacity constraints on the lines, as they are fundamental for various applications.

3.2 AC Power Flow Behavior

Before presenting the models, it is useful to review the behavior of AC power flows, which is the primary motivation of these models. The high-level behavior of power systems is often characterized by two rules of thumb in the literature: (1) phase angles are the primary factor in determining the flow active power; (2) differences in voltage magnitudes are the primary factor in determining the flow of reactive power [16]. Let’s examine these properties with a simple experiment.

This experiment makes two basic assumptions: (1) In the per unit system, voltages do not vary far from a magnitude of 1.0 and an angle of 0.0; (2) The magnitude of a line conductance is much smaller than the magnitude of the susceptance, i.e., \(|g| \ll |b|\). We can then explore the value of the power flow equations (5) and (6), when the voltages are in the following bounds: \(|\tilde{V}_n| = 1.0, |\tilde{V}_m| \in (1.2, 0.8), \theta_n - \theta_m \in (-\pi/6, \pi/6)\). These bounds are intentionally generous so that the power flow behavior within and outside normal operating conditions may be illustrated.

Figure 1 presents the contour of the active power (left) and reactive power (right) equations for a line \((n, m)\) under these assumptions when \(\tilde{Y}_{nm} = 0.2 - i1\). The contour lines indicate significant changes in power flow. Consider first the active power plot (left). For a fixed voltage, varying the phase angle difference (y-axis) induces significant changes in active power as many lines are crossed. In contrast, for a fixed phase angle difference, varying the voltage (x-axis) has limited impact on the active power, since few lines are crossed. Hence, the plot indicates that phase angle differences are the primary factor of active power flow while voltage differences have only a small effect. The situation is quite different for reactive power (right plot). For a fixed voltage, varying the phase angle difference induces some significant changes in reactive power (around four lines can be crossed). But, if the phase angle difference is fixed, varying voltage induces even more significant changes in reactive power (as many as seven lines may be crossed). Hence, changes in voltages are the primary factor of reactive power flows but the phase angle differences also have a significant influence. As we will see in Section 3.5, both the differences in phase angle and voltage magnitude appear in the approximation of reactive power.

3.3 A Polyhedral Relaxation of Cosine

All of the proposed linear programming approximations are built on an observation that a nice polyhedral relaxation of the cosine function exists in the context of power system design and operation. This section
Figure 1: Power Flow Contour of Active (left) and Reactive (right) Power with $g = 0.2$ and $b = -1$.

Figure 2: A Polyhedral Relaxation of Cosine using 7 Inequalities.

discusses how such a relaxation is constructed. First we notice that by the design and operation of power systems, the phase angle difference on a line is typically very small, for example, $\pm \pi/24$ [43]. Hence, it is reasonable to restrict our attention to the cosine function within the generous domain of $(-\pi/2, \pi/2)$, which has a convex structure similar to a quadratic function (See Figure 2). Within this restricted domain, a polyhedral relaxation of the cosine function can be made by a collection of hyperplanes tangent to the cosine function. Figure 2 illustrates such a polyhedral relaxation using seven tangent hyperplanes within the domain of $(-\pi/2, \pi/2)$. The dark black line shows the cosine function, the dashed lines are the linear inequalities, and the shaded area is the feasible region of the linear system formed by the inequalities, i.e. the polyhedral relaxation.

The polyhedron in Figure 2 can be formalized in the following way: Let $\theta_{\Delta} \in (0, \pi/2)$ be the desired bound on the phase angle difference and let $cs$ be the number of tangent hyperplanes to use in the polyhedron. Notice the tangent line to the cosine function for any point in the domain is defined by,

$$y = -\sin(a)(x - a) + \cos(a) \quad \forall a \in (-\theta_{\Delta}, \theta_{\Delta})$$

If we space the $cs$ hyperplanes evenly along the domain $(-\theta_{\Delta}, \theta_{\Delta})$ then the step size $s$ between each tangent point is $s = 2\theta_{\Delta}/(cs + 1)$ and the following system of inequalities can be used to build the
polyhedral relaxation,
\[
y \geq \cos(\theta_\Delta)
\]
\[
y \leq -\sin(ts - \theta_\Delta)(x - ts + \theta_\Delta) + \cos(ts - \theta_\Delta) \quad \forall t \in \{1, 2, \ldots, cs\}
\]
Throughout the rest of the paper \( y \) will be the decision variable \( \hat{\cos}_{nm} \) and \( x \) will be the phase angle difference, \( \theta_n - \theta_m^o \).

3.4 The Hot-Start LPAC Model

The linear-programming approximation of the AC Power flow equations in a hot-start context is based on three ideas:

1. Voltage magnitudes \( |V_n^h| \) \((n \in N)\) are available from the AC base-point solution;
2. The approximation, \( \sin(x) \approx x \), is very accurate for small phase angle differences;
3. A nice polyhedral relaxation of cosine exists for small phase angle differences

Let \( \hat{\cos}(\theta_n^o - \theta_m^o) \) denote the polyhedral relaxation of \( \cos(\theta_n^o - \theta_m^o) \) for a reasonable phase angle difference bound (as discussed in Section 3.3), then the linear-programming approximation solves the line flow constraints,

\[
\hat{p}_{nm} = |V_n^h|^2 g_{nm} - |V_n^h||V_m^h|g_{nm}\hat{\cos}(\theta_n^o - \theta_m^o) - |V_n^h||V_m^h|b_{nm}\hat{\cos}(\theta_n^o - \theta_m^o) - |V_n^h||V_m^h|g_{nm}(\theta_n^o - \theta_m^o)
\]

\[
\hat{q}_{nm} = -|V_n^h|^2 b_{nm} + |V_n^h||V_m^h|b_{nm}\hat{\cos}(\theta_n^o - \theta_m^o) - |V_n^h||V_m^h|g_{nm}(\theta_n^o - \theta_m^o). \tag{11}
\]

The hot-start model is a linear program that replaces the AC power equations 5-6 by equations 10-11 respectively.

A complete linear program for the hot-start LPAC formulation is presented in Model 1. The inputs to the model are: (1) A power network \( PN = \{N, L, G, s\} \), as defined previously; (2) the voltage magnitudes \( |V^h| \) for the buses; (3) a bound on the maximum phase angle difference \( \theta_\Delta \) and; (4) the number of hyperplanes \( cs \) in the polyhedral relaxation of the cosine function. Constraints (M1.1) model the slack bus, which has a fixed phase angle. Constraints (M1.2) and (M1.3) model KCL on the buses. Like in nonlinear AC power flow models, the KCL constraints are not enforced on the slack bus for both active and reactive power and on voltage-controlled generators for reactive power. Constraints (M1.4) and (M1.5) capture the approximate line flows from equations (10) and (11). Finally, Constraints (M1.6) define a system of inequalities capturing the polyhedral relaxation of cosine in the domain \((-\theta_\Delta, \theta_\Delta)\) using \( cs \) hyperplanes for each line in the network.

To our knowledge, this hot-start model is the first linear formulation that captures the cosine contribution to reactive power. However, fixing the voltage magnitudes \( |V^h| \) in the power flow equations is too restrictive in many applications. We now remove this restriction.

3.5 The Warm-Start LPAC Model

This section derives the warm-start LPAC model, i.e., a linear-programming model of the AC power flow equations for the warm-start context. The warm-start context assumes that some target voltages \( |V^\phi| \) are available for all buses except voltage-controlled generators whose voltage magnitudes \( |V^\phi| \) are already known. The network must operate close (e.g., \( \pm 0.1 \) Volts p.u.) to these target voltages, otherwise the hardware may be damaged or the system’s voltage may collapse.

The warm-start LPAC model is based on two key ideas:

1. The flow of active power is not significantly affected by voltage differences, so the hot-start active power approximation is still applicable. Simply substitute \( |V^h| \) for \( |V^\phi| \).
2. The effects of voltage differences on the flow of reactive power can be captured by a linear approximation of the voltage around the nominal operating point \( |V^\phi| \).

To derive the reactive power approximation in the warm-start LPAC model, let \( \phi \) be the difference between the target voltage and the true value, i.e.,

\[
|\tilde{V}_n| = |\tilde{V}^\phi| + \phi \quad \forall n \in N.
\]
Model 1: The Hot-Start LPAC Model.

Inputs:
- \(P_N = (N, L, G, s)\) - power network
- \(|\tilde{V}^h|\) - voltage magnitudes from an AC base-point solution
- \(\theta_{\Delta}^s\) - bound on the phase angle difference \((\in (0, \pi/2))\)
- \(cs\) - number of hyperplanes in the cosine polyhedron
- \(s = 2\theta_{\Delta}^s/(cs + 1)\) - spacing of the hyperplanes

Variables:
- \(\theta_n^s \in (-\infty, \infty)\) - phase angle on bus \(n\) (radians)
- \(\tilde{\cos}_{nm} \in (\cos(\theta_{\Delta}^s), 1)\) - approximation of \(\cos(\theta_n^s - \theta_m^s)\)

Subject to:
- \(\theta_n^s = 0\) (M1.1)
- \(p_n = \sum_{(n,m) \in L} \tilde{p}_{nm}^h \quad \forall n \in N, n \neq s\) (M1.2)
- \(q_n = \sum_{(n,m) \in L} \tilde{q}_{nm}^h \quad \forall n \in N, n \neq s, n \notin G\) (M1.3)
- \(\forall (n, m), (m, n) \in L\)
  \[\tilde{p}_{nm}^h = |\tilde{V}_n^h|^2 g_{nm} - |\tilde{V}_m^h|^2 |g_{nm}\tilde{\cos}_{nm} + b_{nm}(\theta_n^s - \theta_m^s)|\] (M1.4)
  \[\tilde{q}_{nm}^h = -|\tilde{V}_n^h|^2 b_{nm} - |\tilde{V}_m^h|^2 |g_{nm}(\theta_n^s - \theta_m^s) - b_{nm}\tilde{\cos}_{nm}|\] (M1.5)
  \[\tilde{\cos}_{nm} \leq -\sin(ts - \bar{\theta}_{\Delta})(\theta_n^s - \theta_m^s) - ts + \bar{\theta}_{\Delta} + \cos(ts - \bar{\theta}_{\Delta}) \quad \forall t \in \{1, 2, \ldots, cs\}\] (M1.6)

Substituting in equation 6, we obtain
\[q_{nm} = -(|\tilde{V}_n^1|^2 + 2|\tilde{V}_n^1|\phi_n + \phi_n^2)b_{nm} - (|\tilde{V}_m^1||\tilde{V}_n^1| + |\tilde{V}_m^1|\phi_n + \phi_n\phi_m)(g_{nm}\cos(\theta_n^s - \theta_m^s) - b_{nm}\cos(\theta_n^s - \theta_m^s))\] (12)

We can divide this expression into two parts
\[q_{nm} = q_{nm}^t + q_{nm}^\Delta\] (13)

where \(q_{nm}^t\) is equation 6 with \(|\tilde{V}| = |\tilde{V}^t|\) and \(q_{nm}^\Delta\) captures the remaining terms, i.e.,
\[q_{nm}^\Delta = -(2|\tilde{V}_n^1|\phi_n + \phi_n^2)b_{nm} - (|\tilde{V}_m^1|\phi_n + \phi_n\phi_m)(g_{nm}\sin(\theta_n^s - \theta_m^s) - b_{nm}\cos(\theta_n^s - \theta_m^s))\] (14)

Equation 13 is equivalent to equation 6 and must be linearized to obtain the LPAC model.

The \(q_{nm}^t\) component involves constant target voltages and may thus be approximated by \(q_{nm}^t\) (from Section 3.4). The \(q_{nm}^\Delta\) is more challenging as it contains nonlinear and non-convex terms such as \(\phi_n\phi_m\cos(\theta_n^s - \theta_m^s)\). We approximate \(q_{nm}^\Delta\) using the linear terms of the Taylor series of \(q_{nm}^\Delta\) at \(\phi_n = 0, \phi_m = 0, \theta_n^s - \theta_m^s = 0\) to obtain
\[q_{nm}^\Delta = -|\tilde{V}_n^1|b_{nm}(\phi_n - \phi_m) - (|\tilde{V}_m^1| - |\tilde{V}_m^1|)b_{nm}\phi_n\] (15)

A complete linear program for the warm-start LPAC formulation is presented in Model 2. The inputs to the model are similar to Model 1, with hot startup voltages \(|\tilde{V}^h|\) replaced by target voltages \(|\tilde{V}^t|\). Constraints (M2.1) model the slack bus, which has a fixed voltage and phase angle. Constraints (M2.2) capture the voltage-controlled generators which, by definition, do not vary from their voltage target \(|\tilde{V}^t|\). Constraints (M2.3) and (M2.4) model the buses, as well as the effects of voltage change presented in equation (13). Like in AC power flow models, the KCL constraints are not enforced on the slack bus for both active and reactive power and on voltage-controlled generators for reactive power. Constraints (M2.5) and (M2.6) capture the approximate line flows from equations (10) and (11). Constraints (M2.7) model the effects of voltage change presented in equation (15). Finally, Constraints (M2.8) define a system of inequalities capturing the polyhedral relaxation of cosine in the domain \((-\bar{\theta}_{\Delta}, \bar{\theta}_{\Delta})\) using \(cs\) hyperplanes for each line in the network.
Model 2 The Warm-Start LPAC Model.

Inputs:
- \( \mathcal{P} \mathcal{N} = (N, L, G, s) \) - the power network
- |\( \tilde{V}^i \)| - target voltage magnitude
- |\( \tilde{s}^\Delta \) - bound on the phase angle difference (\( (0, \pi/2) \))
- \( cs \) - the number of hyperplanes in the cosine polyhedron
- \( s = 2s^\Delta/(cs + 1) \) - spacing of the hyperplanes

Variables:
- \( \theta^s_n \in (-\infty, \infty) \) - phase angle on bus \( n \) (radians)
- \( \phi_n \in (-|\tilde{V}^i|, \infty) \) - voltage change on bus \( n \) (Volts p.u.)
- \( \cos_{nm} \in (\cos(\tilde{s}^\Delta), 1) \) - Approximation of \( \cos(\theta^s_n - \theta^s_m) \)

Subject to:
- \( \theta^s_n = 0, \phi_n = 0 \) \hspace{5cm} (M2.1)
- \( \phi_i = 0 \forall i \in G \) \hspace{5cm} (M2.2)
- \( p_n = \sum_{(n,m) \in L} \tilde{p}^i_{nm} \forall n \in N \ n \neq s \) \hspace{5cm} (M2.3)
- \( q_n = \sum_{(n,m) \in L} \tilde{q}^i_{nm} + \tilde{\Delta^\Delta}_{nm} \forall n \in N \ n \neq s \ n \notin G \) \hspace{5cm} (M2.4)
- \( \forall (n,m), (m,n) \in L \)
  - \( \tilde{p}^i_{nm} = |\tilde{V}^i_n| |\tilde{V}^i_m| \cos_{nm} + b_{nm}(\theta^s_n - \theta^s_m) \) \hspace{5cm} (M2.5)
  - \( \tilde{q}^i_{nm} = -|\tilde{V}^i_n|^2 b_{nm} - |\tilde{V}^i_m|^2 + (g_{nm}(\theta^s_n - \theta^s_m) - b_{nm} \cos_{nm}) \) \hspace{5cm} (M2.6)
  - \( \tilde{\Delta^\Delta}_{nm} = -|\tilde{V}^i_n|[\phi_n - \phi_m] - (|\tilde{V}^i_n|^2 - |\tilde{V}^i_m|^2) b_{nm} \phi_n \) \hspace{5cm} (M2.7)
  - \( \cos_{nm} \leq -\sin(ts - \tilde{s}^\Delta)(\theta^s_n - \theta^s_m) - ts + \tilde{s}^\Delta \) \hspace{5cm} (M2.8)

3.6 The Cold-Start LPAC Model

In a cold-start context, no target voltages are available and voltage magnitudes are approximated by 1.0, except for voltage-controlled generators whose voltages \( (|\tilde{V}^i_n|) \) are known a priori. The cold-start LPAC model is then derived from the warm-start LPAC model by fixing \( |\tilde{V}^i_n| = 1.0 \) for all \( i \in N \), in which case equation (M2.7) becomes

\[
\tilde{\Delta^\Delta}_{nm} = -b_{nm}(\phi_n - \phi_m) \tag{16}
\]

Similar to the warm-start model, the cold-start model uses the \( \phi_i \)'s to fix the voltage magnitudes of generators to the correct value (i.e. \( \phi_n = |\tilde{V}^i_n| - 1.0 \ n \in G \)).

3.7 Extensions to the LPAC Model

So far, the LPAC models have been presented as feasibility problems. However, it is important to realize that these LPAC models are building blocks that will be embedded into optimization problems with additional side constraints. In other words, their key purpose is to compute accurate approximations of real and reactive power for use in optimization models exploiting them. This section reviews how to generalize the LPAC models with several side constraints for applications in disaster management, reactive voltage support, transmission planning, and vulnerability analysis. The extensions are illustrated on the warm-start model but can be used in the hot-start and cold-start models as well.

Generators The LPAC model can easily be generalized to include ranges for generators: Simply remove the generator from \( G \) and place operating limits on the \( p \) and \( q \) variables for that bus.

Removing the Slack Bus By necessity, AC solvers use a slack bus to ensure the flow balance in the network when the total power consumption is not known a priori (e.g., due to line losses). As a consequence, all of the LPAC models depicted here also use a slack bus, so that the AC and LPAC models can be accurately compared in our experimental results. However, it is important to emphasize that the LPAC model does not need a slack bus and the only reason to include a slack bus in this model is...
to allow for meaningful comparisons between the LPAC and AC models. As discussed above, the LPAC model can easily include a range for each generator, thus removing the need for a slack bus.

**Load Shedding** For applications in power restoration (e.g., [8, 57, 9]), the LPAC model can also integrate load shedding: Simply transform the loads into decision variables with an upper bound. Section 5.1 reports experimental results on such a power restoration model.

**Modeling Additional Side Constraints** In practice, feasibility constraints may exist on the acceptable voltage range, the reactive injection of a generator, or line flow capacities. Because Model 2 is a linear program, it can easily incorporate such constraints. For instance, constraint

$$|V_n| \leq |V_n'| + \phi_n \forall n \in N$$

ensures that voltages are above a certain lower bound $|V_n'|$, constraint

$$\sum_{(n,m) \in L} q_{nm}^i + q_{nm}^\Delta \leq \bar{q}_n \forall n \in G$$

limits the maximum reactive injection bounds at bus $n$ to $\bar{q}_n$. Finally, let $|S_{nm}|$ be the maximum apparent power on a line from bus $n$ to bus $m$. Then, constraint

$$(\dot{P}_{nm})^2 + (q_{nm}^i + q_{nm}^\Delta)^2 \leq |S_{nm}|^2$$

ensures that line flows are feasible in the LPAC model. The quadratic functions can be approximated by a polyhedral outer approximation (e.g., [10]).

**4 Accuracy of the LPAC Model**

This section evaluates the accuracy of the LPAC models by comparing them to an ideal nonlinear AC power flow. The experiments were performed on nine traditional power-system benchmarks which come from the IEEE test systems [56] and MATPOWER [62]. The AC power flow equations were solved with a Newton-Raphson solver which was validated using MATPOWER. The LPAC models use 20 hyperplanes over the range $\pm \pi/3$ in the cosine polyhedral relaxation and all of the models are solved in less than 1 second on a 2.5 GHz Intel processor. The results also include a modified version of the IEEEEd17 benchmark, called IEEEEd17m. The original IEEEEd17 has the slack bus connected to the network by a transformer with $\tilde{T} = 1.05 \angle 0$. The nonlinear behavior of transformers induces some loss of accuracy in the LPAC model and, because this error occurs at the slack bus in IEEEEd17, it affects all buses in the network. IEEEEd17m resolves this issue by setting $\tilde{T} = 1.00 \angle 0$ and the slack bus voltage to 1.05. This equivalent formulation is significantly more accurate for the LPAC model. In these load-flow studies, an objective function maximizing $\sum_{(n,m) \in L} \cos \angle_{nm}$ is added to the LPAC models to select the ideal value in the convex hull of the cosine function. In other applications, maximization of the cosine variables can be a lower order term in a lexicographic objective.

These results provide empirical evaluations of the DC and LPAC models in cold-start and warm-start contexts. They report aggregate statistics for active power (Table 1), bus phase angles (Table 2), reactive power (Table 3), and voltage magnitudes (Table 4). Data for the DC model is necessarily omitted from Tables 3 and 4 as reactive power and voltages are not captured by the DC model. In each table, two aggregate values are presented: Correlation (corr) and absolute error ($\Delta$). The units of the absolute error are presented in the headings. Both average ($\mu$) and worst-case (max) values are presented. The worst case can often be misleading: For example a very large value may actually be a very small relative quantity. For this reason, the tables show the relative error ($\delta$) of the value selected by the max operator using the arg-max operator. The relative error is a percentage and is unit-less.

---

2For consistency, the LPAC models are extended to include line charging, bus shunts, and transformers, as discussed in Appendix A.

3Some classic objectives such as minimizing generation costs or line losses have a side effect of minimizing the cosine variables. In these cases, they may be omitted from the objective entirely.
Table 1: Accuracy of the LPAC Model: Active Power

<table>
<thead>
<tr>
<th>Benchmark</th>
<th>Corr</th>
<th>Active Power (MW)</th>
<th>$\mu(\Delta)$</th>
<th>max($\Delta$)</th>
<th>$\delta(\text{arg-max}(\Delta))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>The DC Model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ieeed17</td>
<td>0.9994</td>
<td>1.392</td>
<td>10.64</td>
<td>6.783</td>
<td></td>
</tr>
<tr>
<td>mp24</td>
<td>0.9989</td>
<td>5.659</td>
<td>19.7</td>
<td>23.65</td>
<td></td>
</tr>
<tr>
<td>ieeed30</td>
<td>0.9993</td>
<td>1.046</td>
<td>13.1</td>
<td>7.562</td>
<td></td>
</tr>
<tr>
<td>mp30</td>
<td>0.9993</td>
<td>0.2964</td>
<td>2.108</td>
<td>19.36</td>
<td></td>
</tr>
<tr>
<td>ieeed17m</td>
<td>0.999</td>
<td>7.341</td>
<td>43.64</td>
<td>6.527</td>
<td></td>
</tr>
<tr>
<td>ieeed57</td>
<td>0.9989</td>
<td>1.494</td>
<td>8.216</td>
<td>8.055</td>
<td></td>
</tr>
<tr>
<td>ieeed118</td>
<td>0.9963</td>
<td>3.984</td>
<td>56.3</td>
<td>44.74</td>
<td></td>
</tr>
<tr>
<td>ieeed117</td>
<td>0.9972</td>
<td>4.933</td>
<td>201.3</td>
<td>13.84</td>
<td></td>
</tr>
<tr>
<td>ieeed17m</td>
<td>0.9975</td>
<td>4.779</td>
<td>191.1</td>
<td>13.23</td>
<td></td>
</tr>
<tr>
<td>mp300</td>
<td>0.9910</td>
<td>11.09</td>
<td>418.5</td>
<td>90.02</td>
<td></td>
</tr>
<tr>
<td>The LPAC-Cold Model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ieeed17</td>
<td>0.9989</td>
<td>1.636</td>
<td>5.787</td>
<td>13.13</td>
<td></td>
</tr>
<tr>
<td>mp24</td>
<td>0.9999</td>
<td>1.884</td>
<td>6.159</td>
<td>2.933</td>
<td></td>
</tr>
<tr>
<td>ieeed30</td>
<td>0.998</td>
<td>0.5475</td>
<td>2.213</td>
<td>2.523</td>
<td></td>
</tr>
<tr>
<td>mp30</td>
<td>0.9995</td>
<td>0.2396</td>
<td>1.641</td>
<td>15.07</td>
<td></td>
</tr>
<tr>
<td>ieeed17</td>
<td>1.0000</td>
<td>2.142</td>
<td>8.043</td>
<td>3.288</td>
<td></td>
</tr>
<tr>
<td>ieeed57</td>
<td>0.9995</td>
<td>0.9235</td>
<td>4.674</td>
<td>9.728</td>
<td></td>
</tr>
<tr>
<td>ieeed118</td>
<td>1.0000</td>
<td>0.622</td>
<td>3.708</td>
<td>2.036</td>
<td></td>
</tr>
<tr>
<td>ieeed17m</td>
<td>0.9999</td>
<td>1.837</td>
<td>30.38</td>
<td>2.088</td>
<td></td>
</tr>
<tr>
<td>ieeed17m</td>
<td>0.9999</td>
<td>1.475</td>
<td>20.21</td>
<td>1.399</td>
<td></td>
</tr>
<tr>
<td>mp300</td>
<td>0.9998</td>
<td>2.455</td>
<td>18</td>
<td>8.675</td>
<td></td>
</tr>
<tr>
<td>The LPAC-Warm Model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ieeed17</td>
<td>1.0000</td>
<td>0.1689</td>
<td>1.588</td>
<td>1.012</td>
<td></td>
</tr>
<tr>
<td>mp24</td>
<td>1.0000</td>
<td>0.6621</td>
<td>2.041</td>
<td>1.01</td>
<td></td>
</tr>
<tr>
<td>ieeed30</td>
<td>1.0000</td>
<td>0.1847</td>
<td>2.333</td>
<td>1.405</td>
<td></td>
</tr>
<tr>
<td>mp30</td>
<td>0.9999</td>
<td>0.1052</td>
<td>0.705</td>
<td>6.474</td>
<td></td>
</tr>
<tr>
<td>ieeed17</td>
<td>1.0000</td>
<td>1.557</td>
<td>11.58</td>
<td>1.731</td>
<td></td>
</tr>
<tr>
<td>ieeed118</td>
<td>0.9999</td>
<td>0.4386</td>
<td>7.376</td>
<td>5.862</td>
<td></td>
</tr>
<tr>
<td>ieeed17m</td>
<td>1.0000</td>
<td>0.58</td>
<td>22.5</td>
<td>1.547</td>
<td></td>
</tr>
<tr>
<td>ieeed17m</td>
<td>1.0000</td>
<td>0.5725</td>
<td>21.73</td>
<td>1.504</td>
<td></td>
</tr>
<tr>
<td>mp300</td>
<td>0.9999</td>
<td>1.135</td>
<td>52.84</td>
<td>11.57</td>
<td></td>
</tr>
</tbody>
</table>

Table 2 indicates uniform improvements in line active power flows ($p_{nm}$), especially in the largest benchmarks IEEEd118, IEEEd17, and MP300. Significant errors are not uncommon for the DC model on large benchmarks [52] and are primarily caused by line losses which are not modelled. Due to its asymmetrical power flow equations and the cosine relaxation, the LPAC model does not suffer from these shortcomings.

Table 2 presents the aggregate statistics on bus phase angles ($\theta_n$). These results show significant improvements in accuracy especially on larger benchmarks. The correlations are somewhat lower than active power, but phase angles are quite challenging from a numerical accuracy standpoint.

Table 3 presents the aggregate statistics on line reactive power flows ($q_{nm}$). They indicate that reactive power flows are generally accurate and highly precise in warm-start contexts. To highlight the model accuracy in cold-start contexts, the reactive flow correlation for the two worst benchmarks, IEEEd17m and MP300, is presented in Figure 3.

Table 4 presents the aggregate statistics on bus voltage magnitudes ($|V_n|$). These results indicate that voltage magnitudes are very accurate on small benchmarks, but the accuracy reduces with the size of the network. The warm-start context brings a significant increase in accuracy in larger benchmarks. To illustrate the quality of these solutions in cold-start contexts, the voltage magnitude correlation for the two worst benchmarks, i.e., IEEEd17m and MP300, is presented in Figure 4. We notice that the increase in voltage errors is related to the distance from a load point to the nearest generator. The linearized voltage model incurs some small error on each line. As the voltage changes over many lines,
Table 3: Accuracy of the LPAC Model: Reactive Power Flow.

<table>
<thead>
<tr>
<th>Benchmark</th>
<th>Corr</th>
<th>Reactive Power (MVar)</th>
<th>$\mu(\Delta)$</th>
<th>$\max(\Delta)$</th>
<th>$\delta(\text{arg-max}(\Delta))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>The LPAC-Cold Model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ieee14</td>
<td>0.9948</td>
<td>0.7459</td>
<td>2.561</td>
<td>14.92</td>
<td></td>
</tr>
<tr>
<td>mp24</td>
<td>0.9992</td>
<td>1.505</td>
<td>5.245</td>
<td>9.309</td>
<td></td>
</tr>
<tr>
<td>ieee30</td>
<td>0.997</td>
<td>0.4962</td>
<td>2.336</td>
<td>9.309</td>
<td></td>
</tr>
<tr>
<td>mp30</td>
<td>0.9991</td>
<td>0.3135</td>
<td>3.866</td>
<td>3.866</td>
<td></td>
</tr>
<tr>
<td>mp39</td>
<td>0.9973</td>
<td>3.898</td>
<td>15.15</td>
<td>18.25</td>
<td></td>
</tr>
<tr>
<td>ieee57</td>
<td>0.9991</td>
<td>0.5316</td>
<td>2.98</td>
<td>2.98</td>
<td></td>
</tr>
<tr>
<td>ieee118</td>
<td>0.9991</td>
<td>0.7676</td>
<td>6.248</td>
<td>8.561</td>
<td></td>
</tr>
</tbody>
</table>

| The LPAC-Warm Model | | | | | |
| ieee14 | 0.9895 | 0.8689 | 3.167 | 43.89 |
| mp24 | 0.9992 | 1.505 | 5.245 | 9.309 |
| ieee30 | 0.997 | 0.3455 | 1.607 | 7.62 |
| mp30 | 0.9991 | 0.3135 | 3.866 | 3.866 |
| mp39 | 0.9971 | 4.03 | 15.75 | 18.25 |
| ieee57 | 0.9995 | 0.3853 | 1.46 | 5.67 |
| ieee118 | 0.9991 | 0.7676 | 6.248 | 8.561 |
| ieee118m | 0.9979 | 3.898 | 48.65 | 40.58 |
| mp300 | 0.9981 | 3.85 | 62.32 | 17.98 |

Table 4: Accuracy of the LPAC Model: Voltage Magnitudes.

<table>
<thead>
<tr>
<th>Benchmark</th>
<th>Corr</th>
<th>Voltage Magnitude (Volts p.u.)</th>
<th>$\mu(\Delta)$</th>
<th>$\max(\Delta)$</th>
<th>$\delta(\text{arg-max}(\Delta))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>The LPAC-Cold Model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ieee14</td>
<td>0.9828</td>
<td>0.003524</td>
<td>0.01304</td>
<td>1.236</td>
<td></td>
</tr>
<tr>
<td>mp24</td>
<td>0.9983</td>
<td>0.000676</td>
<td>0.003244</td>
<td>0.3362</td>
<td></td>
</tr>
<tr>
<td>ieee30</td>
<td>0.9908</td>
<td>0.002445</td>
<td>0.01098</td>
<td>1.236</td>
<td></td>
</tr>
<tr>
<td>mp30</td>
<td>0.9884</td>
<td>0.002186</td>
<td>0.009453</td>
<td>0.9723</td>
<td></td>
</tr>
<tr>
<td>mp39</td>
<td>0.9972</td>
<td>0.3135</td>
<td>0.8925</td>
<td>3.886</td>
<td></td>
</tr>
<tr>
<td>ieee57</td>
<td>0.9976</td>
<td>0.3455</td>
<td>1.607</td>
<td>7.62</td>
<td></td>
</tr>
<tr>
<td>ieee118</td>
<td>0.9991</td>
<td>0.7676</td>
<td>6.248</td>
<td>8.561</td>
<td></td>
</tr>
</tbody>
</table>

| The LPAC-Warm Model | | | | | |
| ieee14 | 0.998 | 0.0005479 | 0.001173 | 0.1111 |
| mp24 | 0.9986 | 0.000542 | 0.002214 | 0.2294 |
| ieee30 | 0.9994 | 0.001426 | 0.002508 | 0.25 |
| mp30 | 1.000 | 0.0003884 | 0.000707 | 0.073 |
| mp39 | 0.9983 | 0.005154 | 0.003545 | 0.3524 |
| ieee57 | 0.9987 | 0.002138 | 0.005315 | 0.59 |
| ieee118 | 0.9999 | 0.0001961 | 0.001303 | 0.1344 |
| ieee118m | 0.9979 | 3.898 | 48.65 | 40.58 |
| mp300 | 0.9981 | 3.85 | 62.32 | 17.98 |

Table 5: Percentage of Voltage-Controlled Buses in the Benchmarks.

<table>
<thead>
<tr>
<th>Benchmark</th>
<th>Percentage of Voltage-Controlled Buses</th>
</tr>
</thead>
<tbody>
<tr>
<td>ieee14</td>
<td>35.7%</td>
</tr>
<tr>
<td>mp24</td>
<td>45.8%</td>
</tr>
<tr>
<td>ieee30</td>
<td>20.0%</td>
</tr>
<tr>
<td>mp30</td>
<td>20.0%</td>
</tr>
<tr>
<td>mp39</td>
<td>30.8%</td>
</tr>
<tr>
<td>ieee57</td>
<td>12.3%</td>
</tr>
<tr>
<td>ieee118</td>
<td>45.8%</td>
</tr>
<tr>
<td>ieee118m</td>
<td>7.4%</td>
</tr>
<tr>
<td>mp300</td>
<td>24.7%</td>
</tr>
</tbody>
</table>

these small errors accumulate. By comparing the percentage of voltage-controlled generator buses in each benchmark $|G|/|N|$ (Table 5) to the accuracy values in Table 4, we can see from the IEEE57 and IEEEEd17 benchmarks that a low percentage is a reasonable indicator of the voltage accuracy in the cold-start context.

4.1 Alternative Linear Models

The formulation of the LPAC models explicitly removes two core assumptions of the traditional DC model:
1. Although $\cos(\theta_n - \theta_m)$ maybe very close to 1, those small deviations are important.
2. Although $|g| \ll |b|$, the conductance contributes significantly to the phase angles and voltage magnitudes.

This section investigates three variants of the cold-start LPAC model that reintegrate some of the assumptions of the DC model. The new models are: (1) the LPAC-C model where only the cosine relaxation is used and $g = 0$; (2) the LPAC-G model where only the $g$ value is used and $\cos(x) = 1$; (3) the LPAC-CG model where $\cos(x) = 1$ and $g = 0$. Tables 6 and 7 present the cumulative absolute error between the proposed linear formulations and the true nonlinear solutions. Many metrics may be of interest but these results focus on line voltage drop $\tilde{\mathcal{V}}_n - \tilde{\mathcal{V}}_m$ and bus power $\tilde{S}_n$. These were selected because they are robust to errors which accumulate as power flows through the network. The results highlight two interesting points. First, all LPAC based models bring improvements over a traditional DC model. Second, although integrating either the $g$ value or the cosine term brings some small improvement independently, together they make significant improvements in accuracy. Additionally a comparison of Table 6 and Table 7 reveals that the benefits of the new linear models are more pronounced as the network size increases.
Figure 3: Reactive Power Flow Correlation for the LPAC Model on IEEEdd17m (left) and MP300 (right) in a Cold-start Context.

Figure 4: Voltage Magnitude Correlation for the LPAC Model on IEEEdd17m (left) and MP300 (right) in a Cold-start Context.

5 Case Studies

This section describes two case studies to evaluate the potential of the LPAC models to decision-support applications in power restoration and capacitor placement. The goal is not to present comprehensive solutions for these two complex problems, but to provide evidence that the LPAC models strikes an appealing compromise between efficiency and accuracy for such applications. They indicate that LPAC models, together with state-of-the-art MIP solvers, provide an attractive solution for applications where the DC model is not accurate enough and existing approaches are too time consuming or suboptimal.

5.1 Power Restoration

After a significant disruption (e.g., a natural disaster) large sections of the power network need to be re-energized. To understand the effects of restoration actions, power engineers must simulate the network behavior under various courses of action. However, the network is far from its normal operating state, which makes it extremely challenging to solve the AC power flow equations. In fact, the task of finding an AC solution without a reasonable starting point has been regarded as “maddeningly difficult” [39].
Table 6: Accuracy Comparison of Linear Models (Part I).

<table>
<thead>
<tr>
<th>Model</th>
<th>Cumulative Absolute Error</th>
<th>ℜ(\tilde{V}_n - \tilde{V}_m)</th>
<th>ℑ(\tilde{V}_n - \tilde{V}_m)</th>
<th>\text{pn}</th>
<th>\text{qn}</th>
</tr>
</thead>
<tbody>
<tr>
<td>DC</td>
<td>0.3839</td>
<td>0.166</td>
<td>13.39</td>
<td>118.4</td>
<td></td>
</tr>
<tr>
<td>LPAC-A-GC</td>
<td>0.1561</td>
<td>0.1296</td>
<td>13.39</td>
<td>140.7</td>
<td></td>
</tr>
<tr>
<td>LPAC-A-G</td>
<td>0.1221</td>
<td>0.1229</td>
<td>13.39</td>
<td>120.7</td>
<td></td>
</tr>
<tr>
<td>LPAC-A-C</td>
<td>0.1277</td>
<td>0.1262</td>
<td>8.843</td>
<td>53.24</td>
<td></td>
</tr>
<tr>
<td>LPAC</td>
<td>0.1008</td>
<td>0.1234</td>
<td>1.783</td>
<td>11.43</td>
<td></td>
</tr>
<tr>
<td>mp24</td>
<td>DC</td>
<td>0.448</td>
<td>0.1434</td>
<td>53.22</td>
<td>792.4</td>
</tr>
<tr>
<td></td>
<td>LPAC-A-GC</td>
<td>0.3676</td>
<td>0.1289</td>
<td>53.22</td>
<td>546.2</td>
</tr>
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<td>LPAC-A-G</td>
<td>0.2309</td>
<td>0.1332</td>
<td>31.43</td>
<td></td>
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<tr>
<td></td>
<td>LPAC-A-C</td>
<td>0.2417</td>
<td>0.116</td>
<td>47.62</td>
<td></td>
</tr>
<tr>
<td></td>
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</tr>
<tr>
<td></td>
<td>LPAC</td>
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<td>0.5252</td>
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<td>142.1</td>
</tr>
</tbody>
</table>

Table 7: Accuracy Comparison of Linear Models (Part II).

<table>
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<tr>
<th>Model</th>
<th>Cumulative Absolute Error</th>
<th>ℜ(\tilde{V}_n - \tilde{V}_m)</th>
<th>ℑ(\tilde{V}_n - \tilde{V}_m)</th>
<th>\text{pn}</th>
<th>\text{qn}</th>
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<td>0.6803</td>
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<td>LPAC</td>
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<td>0.6803</td>
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<tr>
<td></td>
<td>LPAC-A-GC</td>
<td>0.7298</td>
<td>0.9944</td>
<td>132.7</td>
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<td>LPAC-A-G</td>
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<td>LPAC</td>
<td>0.2625</td>
<td>0.5252</td>
<td>72.27</td>
<td>142.1</td>
</tr>
</tbody>
</table>

The LPAC model studied here has the benefit of providing starting values for all the variables in the AC power flow problem, unlike the traditional DC which only provides active power values. Furthermore, the LPAC model has the additional advantage of supporting bounds on reactive generation and voltage magnitudes and such constraints are critical for providing feasible solutions to the AC power flow. This section illustrates these benefits.

Before presenting the power-restoration model, it is important to mention the key aspect of this application. When the power system undergoes significant damages, load shedding must occur. The DC and LPAC models must be embedded in a restoration model that maximizes the served load given operational constraints such as the generation limits. These load values indicate the maximum amount of power that can be dispatched while ensuring system stability. Model 3 presents a linear program based on the warm-start LPAC model which, given limits on active power generation \( p^g \) and the desired active and reactive loads \( p^l \) and \( q^l \) at each bus, determines the maximum amount of load that can be dispatched. The model assumes that the loads can be shed continuously and that the active and reactive parts of the load should maintain the same power factor. The objective function (M3.1) is lexicographic, it first maximizes the percentage of load served, followed by maximizing the value of the cosine relaxation. Constraint (M3.2) and (M3.3) set the active and reactive injection at bus \( n \) appropriately based on the decision variables for load shedding and generation dispatch. Constraint (M3.4) ensures that reactive
**Model 3**: A LP for Maximizing Desired Load.

**Inputs:**
- \( p_n \) - maximum active injection for bus \( n \)
- \( \tilde{p}_n \) - desired active load at bus \( n \)
- \( q_n \) - desired reactive load at bus \( n \)

Inputs from Model 2 (The Warm-Start LPAC Model)

**Variables:**
- \( p_n \in (0, \tilde{p}_n) \) - active generation at bus \( n \)
- \( q_n \in (-\infty, \infty) \) - reactive generation at bus \( n \)
- \( l_n \in (0, 1) \) - percentage of load served at bus \( n \)

Variables from Model 2 (The Warm-Start LPAC Model)

**Maximize:**
\[
\sum_{n \in N} \tilde{p}_n l_n, \sum_{(n,m) \in L} \bar{c} \bar{s}_{nm} \quad (M3.1)
\]

**Subject to:**
\[
p_n = -p_n l_n + \tilde{p}_n \quad \forall n \in N \quad (M3.2)
\]
\[
q_n = -q_n l_n + q_n \quad \forall n \in N \quad (M3.3)
\]
\[
q_n = 0 \quad \forall n \in N \setminus G \quad (M3.4)
\]
\[
q_n = \sum_{(n,m) \in L} \tilde{q}_{nm} + \delta_{nm} \quad \forall n \in G \quad (M3.5)
\]

Constraints from Model 2 (The Warm-Start LPAC Model)

generation only occurs at generator buses and constraint \((M3.5)\) now defines \( q_n \) for generator buses as well.

Since it reasons about reactive power and voltage magnitudes, Model 3 can be further enhanced to impose bounds on these values. As we will show, such bounds are often critical to obtain high-quality solutions in power restoration contexts. If a reactive generation bound \( \tilde{q}^r \) is supplied, this model can be extended by adding the constraint,
\[
q_n^0 \leq \tilde{q}^r_n \quad \forall n \in G.
\]
Voltage magnitude limits can also be incorporated. Given upper and lower voltage limits \( |\tilde{V}_n| \) and \( |\bar{V}_n| \), the constraint,
\[
|\bar{V}_n| \leq |\tilde{V}_n| + \phi_n \leq |\bar{V}_n| \quad \forall n \in N.
\]
may be used to enforce bounds on voltage magnitudes. The experimental results study the benefits of the LPAC model, suitably enhanced to capture these extensions, for power restoration. They compare a variety of linear models including the DC model, the LPAC model, and enhancements of the LPAC model with additional constraints on reactive power and voltage magnitudes.

Table 8 studies the applicability of various linear power models for network restoration on the IEEE30 benchmark. 1000 line outage cases were randomly sampled from each of the \( N−3, N−4, N−5, \ldots, N−20 \) contingencies. Each contingency is solved with a linear power model (e.g., the DC model or the LPAC model), whose solution is then used as a starting point for the AC model. The performance metric is the number of cases where the AC solver converges, as a good linear model should yield a feasible generation dispatch and a good starting point for the AC solver. To understand the importance of various network constraints, four linear models are studied: the traditional DC model; the LPAC model; the LPAC model with constraints on reactive generation (LPAC-R); and the LPAC with constraints on reactive generation and voltage limits (LPAC-R-V). The number of solved models for each of the contingency classes is presented in Table 8. The results indicate that a traditional DC model is overly optimistic and often produces power dispatches that do not lead to feasible AC power flows (the \( N−10 \) and \( N−13 \) are particularly striking). In contrast, each refinement of the LPAC model solves more contingencies. The LPAC-R-V model is very reliable and is able to produce feasible dispatches in all but 40 contingencies. This means that the LPAC-R-V model solves 99.76% of the 17,000 contingencies studied. Table 9 depicts the load shed by the various models. For large contingencies, the LPAC-R-V model not only provides good starting points for an AC solver but its load shedding is only slightly larger than the (overly
optimal) DC model. These results provide compelling evidence of the benefits of the LPAC model for applications dealing with situations outside the normal operating conditions. In addition, Model 3 can replace the DC model in power restoration applications (e.g., [8, 57]) that are using MIP models to minimize the size of a blackout over time.

### 5.2 Capacitor Placement

The Capacitor Placement Problem (CPP) is a well-studied application [1, 12, 19] and many variants of the problem exist. This section uses a simple version of the problem to demonstrate how the LPAC model can be used as a building block inside a MIP solver for decision-support applications.

Informally speaking, the CPP consists of placing capacitors throughout a power network to improve the voltage profile. The version studied here aims at placing as few capacitors as possible throughout the network, while meeting a lower bound $|\overline{V}|$ on the voltages and satisfying a capacitor injection limit $\overline{q}_c^*$ and reactive generation limits $\overline{q}_{g_n}$ ($n \in G$). Model 4 presents a CPP formulation based on the cold-start LPAC model. For each bus $n$, the additional decision variables are the amount of reactive support added by the capacitor $q_{n}^*$ and a variable $c_n$ indicating whether a capacitor was used.

The objective function (M4.1) is lexicographic, first it minimizes the number of capacitors placed, followed by maximizing the value of the cosine relaxation. Constraints (M4.2) ensure the voltages do not drop below the desired limit and do not exceed the preferred operating condition of 1.05 Volts p.u. Constraints (M4.3) link the capacitor injection variables with the indicator variables. Constraints (M4.4) ensure each generator $n \in G$ does not exceed its reactive generation limit $\overline{q}_{g_n}$, and constraints (M4.5) define the reactive power for generators. Lastly, Constraints (M4.6) redefine the reactive power equation to inject the capacitor contribution $q_{n}^*$. The remainder of the model is the same as the cold-start LPAC model.

The CPP model was tested on a modified version of the IEEE57 benchmark. All of the IEEE benchmarks have sufficient reactive support in their given configurations. To make an interesting capacitor placement problem, the transformer tap ratios are set to 1.0 and existing synchronous condensers are removed. This modified benchmark (IEEE57-C) has significant voltage problems with several bus voltages dropping below 0.9. By design, a solution to Model 4 satisfies all of the desired constraints. However, Model 4 is based on the LPAC model and is only an approximation of the AC power flow. To understand the true value of Model 4, we validate the resulting solution network with an AC solver and measure how

<table>
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<th>LPAC-R-V</th>
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</table>
Model 4 A MIP for the Capacitor Placement Problem.

Inputs:
- \( \overline{q}_n \) - injection bound for generator \( n \in G \)
- \( \overline{q} \) - capacitor injection bound
- \( |\tilde{V}| \) - minimum desired voltage magnitude

Inputs from the Cold-Start LPAC Model

Variables:
- \( q_n \in (0, \overline{q}) \) - capacitor reactive injection
- \( c_n \in \{0, 1\} \) - capacitor placement indicator

Variables from the Cold-Start LPAC Model

Minimize:
\[
\sum_{n \in N} c_n + \sum_{(n,m) \in L} -\tilde{c}_{n,m}
\]  (M4.1)

Subject to:
- \( |\tilde{V}| \leq 1.0 + \phi_n \leq 1.05 \) \( \forall n \in N \)  (M4.2)
- \( q_n \leq \overline{q}_n \) \( \forall n \in G \)  (M4.3)
- \( q_n = \sum_{(n,m) \in L} \tilde{q}_{n,m} + \tilde{q}_{n,m} \) \( \forall n \in G \)  (M4.4)
- \( q_n - \overline{q}_n = \sum_{(n,m) \in L} \tilde{q}_{n,m} + \tilde{q}_{n,m} \) \( \forall n \in N \setminus \{s\} \cup \{\notin G\} \)  (M4.5)

Constraints from the Cold-Start LPAC Model except for the \( q_n \) equations.

---

Table 10: Capacitor Placement: Effects of \( |\tilde{V}| \) on IEEE57-C, \( q^* = 30 \) MVar

| \( |V| \) | \( \min(|V|) \) | \( \max(|V|) \) | \( \max(q_n) \) | \( \sum c_n \) | Time (sec.) |
|---|---|---|---|---|---|
| 0.8850 | 0.000000 | 0.0 | 0.0 | 1 | 1 |
| 0.9350 | 0.000000 | 0.0 | 0.0 | 3 | 8 |
| 0.9600 | 0.000000 | 0.0 | 0.0 | 5 | 156 |
| 0.9750 | 0.000000 | 0.0 | 0.0 | 6 | 177 |
| 0.9775 | 0.000000 | 0.0 | 0.0 | 6 | 139 |
| 0.9800 | 0.000000 | 0.0 | 0.0 | 6 | 75 |
| 0.9840 | -0.000802 | 0.0 | 0.0 | 7 | 340 |

much the constraints are violated. Table 10 presents the results of Model 4 on benchmark IEEE57-C with \( \overline{q} = 30 \) and various thresholds \( |V| \). The table presents the following quantities: The minimum desired voltage \( |\tilde{V}| \); The worst violation of the voltage lower-bound \( \min(|\tilde{V}|) \); The worst violation of the voltage upper bound \( \max(|\tilde{V}|) \); The worst violation of reactive injection upper bound \( \max(q_n) \); The number of capacitors placed \( \sum c_n \); and the runtime of the MIP solver to prove the optimal placement solution. The table indicates that the CPP model is extremely accurate and only has minor constraint violations on the lower bounds of the voltage values. It is important to note that, although the CPP model can take as long as five minutes to prove optimality\(^4\), it often finds the best solution value within a few seconds.

Once again, the CPP model indicates the benefits of the LPAC approximation for decision-support applications that need to reason about reactive power and voltages.

6   Related Work

Several approximations and relaxations of the AC power flow equations have been developed [59, 55, 26, 48, 25, 54, 51]. Broadly, they can be grouped into iterative methods [55, 48, 51] and convex models [59, 26, 25, 54].

\(^4\)It is of course only optimal up the quality of the LPAC approximation.
Iterative Methods Iterative methods, such as the fast-decoupled load flow [51], significantly reduce the computation time of solving the AC equations and demonstrate sufficient accuracy. Their disadvantage however is that they cannot be efficiently integrated into traditional decision-support tools. Indeed, MIP solvers require purely declarative models to obtain lower bounds that are critical in reducing the size of the search space. Note however that, modulo the linear approximation, the LPAC model can be viewed as solving a decoupled load flow globally. The key differences are:

1. Because the model forms one large linear system, all of the steps of the decoupled load flow are effectively solved simultaneously;
2. Because the formulation is a linear program, the values of p and q can now be decision variables, and bounds may be placed on the line capacities, voltage magnitudes, and phase angles;
3. The model may be embedded in a MIP solver for making discrete decisions about the power system.

The second and third points represent significant advantages over the fast-decoupled load flow and other iterative methods.

Convex Models Although many variants of the DC model exist, few declarative models incorporate reactive flows in cold-start contexts. To our knowledge, three cold-start approaches incorporating reactive flows have been proposed: (1) a polynomial approximation scheme [54], (2) a semi-definite programming relaxation [26], and (3) a voltage-difference model [25].

The polynomial approximation has the advantage of solving a convex relaxation of the AC power equations but the number of variables and constraints needed to model the relaxation “grows rapidly” [54] and only second-order terms were considered. The accuracy of this approach for general power flows remains an open question as [54] focuses on a transmission planning application and does not quantify the accuracy of the approximation relative to an AC power flow.

The semi-definite programming (SDP) relaxation [26] has the great advantage that it can solve the power flow equations precisely, without any approximation. In fact, reference [26] demonstrated that the formulation finds the globally optimal value to the AC optimal power flow problem on a number of traditional benchmarks. However, recent work has shown this does not hold on some practical examples [29]. Computationally, SDP solvers are also less mature than LP solvers and their scalability remains an open question [33]. Solvers integrating discrete variables on top of SDP models [30] are very recent and do not have the scientific maturity of MIP solvers [6].

The voltage-difference model [25] has a resemblance to a model combining the equation
\[ \hat{p}_{nm} = \frac{1}{|\tilde{V}_h|^2 - |\tilde{V}_m|^2} (\tilde{V}_h^* \tilde{V}_m^* - \tilde{V}_h \tilde{V}_m) \]
with equation (16). However, it makes a fundamental assumption that all voltages are the same before computing the voltage differences. In practice, voltage-controlled generators violate this assumption. On traditional power system benchmarks, we observed that this voltage-difference formulation had similar accuracy to the DC model.

7 Conclusion This paper presented linear programs to approximate the AC power flow equations. These linear programs, called the LPAC models, capture both the voltage phase angles and magnitudes, which are coupled through equations for active and reactive power. The models use a polyhedral relaxation of cosine term in the power flow equations and the cold-start and warm-start models use a Taylor series for approximating the remaining nonlinear terms.

The LPAC models have been evaluated experimentally on a variety of standard IEEE and MATPOWER benchmarks under normal operating conditions and under contingencies of various sizes. Experimental comparisons with AC solutions demonstrates that the LPAC models are highly accurate for active and reactive power, phase angles, and voltage magnitudes. The paper also presented two case studies in power restoration and capacitor placement to provide evidence that the cold-start and warm-start LPAC models can be efficiently used as a building block for optimization problems involving constraints on reactive power flow and voltage magnitudes. As a result, the LPAC models have the potential to broaden the
success of the traditional DC model into new application areas and to bring increased accuracy and reliability to current DC applications.

There are many opportunities for further study, including the application of the LPAC models to a number of application areas. From an analysis standpoint, it would be interesting to compare the LPAC models with an AC solver using a distributed slack bus. Such an AC solver models the real power systems more accurately and provides a better basis for comparison, since the LPAC models are easily extended to flexible load and generation at all buses.

8 Acknowledgment

The authors would like to thank Mohammad Aldeen, Scott Backhaus, Russell Bent, Misha Chertkov, David Hill, Ian Hiskens, and George Matthews for stimulating discussions and insights on this work. This work was conducted in part at NICTA and is funded by the Australian Government as represented by the Department of Broadband, Communications and the Digital Economy and the Australian Research Council through the ICT Centre of Excellence program.

References


A Extensions for Power Network Benchmarks

For clarity this paper has focused on a simple power line model; where each line is a conductor with an admittance \( \tilde{Y} \). However, the standard benchmarks have additional power flow data including, bus shunts, line charging, and transformers. This appendix introduces these additional parameters and shows how equations (3-6), are extended to support this data.

**Bus Shunts** A bus \( n \in N \) may have a shunt element which is modeled as a fixed admittance to ground with a value of \( \tilde{Y}_{n}^{*} = g_{n}^{*} + ib_{n}^{*} \).

**Line Charging** A line \( \langle n, m \rangle \in L \) may have a predefined line charge \( ib^{c} \). Steady state AC models typically assume that the line charge is evenly distributed across the line and hence it is reasonable to assign equal portions of its charge to both sides of the line. This is incorporated by applying half of the \( ib^{c} \) value to each side of the line power flow, \( \tilde{S}_{nm} \) and \( \tilde{S}_{mn} \).

**Transformers** A transformer \( \langle n, m \rangle \in L \) can be modeled as a line with the additional transformer property, \( T_{nm} = |\tilde{T}_{nm}| \angle s_{nm}^{o} \), where \( |\tilde{T}_{nm}| \) is the tap ratio and \( s_{nm}^{o} \) is the phase shift, both in the direction from \( n \) to \( m \). It is important to note that transformers have an orientation, from \( n \) to \( m \). Hence, transformers modify the power flow \( \tilde{S}_{nm} \) differently than \( \tilde{S}_{mn} \).

**The Complete Flow Equations** Without loss of generality, we will assume all transformers are oriented from from \( n \) to \( m \) with \( n < m \). Let \( L' = \{\langle n, m \rangle \in L : n < m \} \), then the power flow model, i.e., equations (3-6), are extended to support this additional network data as follows:

\[
\begin{align*}
p_{n} &= |\tilde{V}_{n}|^{2}g_{n}^{*} + \sum_{\langle n,m \rangle \in L} p_{nm} \quad \forall \ n \in N \\
q_{n} &= -|\tilde{V}_{n}|^{2}b_{n}^{*} + \sum_{\langle n,m \rangle \in L} q_{nm} \quad \forall \ n \in N \\
p_{nm} &= \frac{|\tilde{V}_{n}|^{2}}{|T_{nm}|^{2}}g_{nm} - \frac{|\tilde{V}_{n}|}{|T_{nm}|} \tilde{V}_{m}(g_{nm} \cos(\theta_{n}^{0} - \theta_{m}^{0} - s_{nm}^{0}) + b_{nm} \sin(\theta_{n}^{0} - \theta_{m}^{0} - s_{nm}^{0})) \quad \forall \langle n, m \rangle \in L' \\
q_{nm} &= -\frac{|\tilde{V}_{n}|^{2}}{|T_{nm}|^{2}}(b_{nm}^{c}/2 + b_{nm}) - \frac{|\tilde{V}_{n}|}{|T_{nm}|} \tilde{V}_{m}(g_{nm} \sin(\theta_{n}^{0} - \theta_{m}^{0} - s_{nm}^{0}) - b_{nm} \cos(\theta_{n}^{0} - \theta_{m}^{0} - s_{nm}^{0})) \quad \forall \langle n, m \rangle \in L' \\
p_{mn} &= |\tilde{V}_{m}|^{2}g_{mn} - |\tilde{V}_{m}| \frac{|\tilde{V}_{n}|}{|T_{nm}|} (g_{mn} \cos(\theta_{m}^{0} - \theta_{n}^{0} + s_{nm}^{0}) + b_{mn} \sin(\theta_{m}^{0} - \theta_{n}^{0} + s_{nm}^{0})) \quad \forall \langle n, m \rangle \in L' \\
q_{mn} &= -|\tilde{V}_{m}|^{2}(b_{mn}^{c}/2 + b_{mn}) - |\tilde{V}_{m}| \frac{|\tilde{V}_{n}|}{|T_{nm}|} (g_{mn} \sin(\theta_{m}^{0} - \theta_{n}^{0} + s_{nm}^{0}) - b_{mn} \cos(\theta_{m}^{0} - \theta_{n}^{0} + s_{nm}^{0})) \quad \forall \langle n, m \rangle \in L' 
\end{align*}
\]

In practice, these equations may be simplified significantly by precomputing the constant values, e.g., \((b_{nm}^{c}/2 + b_{nm})/|T_{nm}|^{2}, g_{nm}/|T_{nm}|\), and so on. With these new constants and the addition of the phase shift term, the equations become a simple variant of equations (3-6).